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THE THEOREMS OF SAALSCHÜTZ AND DOUGALL

By J. L. BURCHNALL and A. LAKIN (*Durham*)

[Received 12 April 1949]

1. In this note we illustrate by two examples a method* which derives the values of certain hypergeometric functions of argument unity directly from the differential equations satisfied by these functions. Although the instances here selected both concern terminating series, the method may, with proper safeguards, be extended to some non-terminating series. For these a final stage in the argument depending upon a limiting process or an appeal to Carlson's theorem is in general necessary; in particular our proof of Dixon's theorem regarding the well-poised ${}_3F_2$ is, though in different dress, essentially that of Bailey.† An interesting feature of the approach adopted here is that the special position of 'well-poised' and 'terminating Saalschützian' series no longer appears an arithmetical accident but flows naturally from the analysis. The method may also be applied *mutatis mutandis* to functions of argument -1.

2. We denote a generalized hypergeometric function by $F(a_r; c_s; x)$ and recall that, if $\delta \equiv x\dfrac{d}{dx}$, then

$$\prod (c_s)\,\delta F = x \prod (a_r) F(a_r+1; c_s+1; x), \tag{1}$$

$$(\delta+a_r)F = a_r F(a_r+1; c_s; x), \tag{2}$$

$$(\delta+c_s-1)F = (c_s-1)F(a_r; c_s-1; x), \tag{3}$$

where in (2) and (3) only the parameter mentioned on the left side is augmented or diminished by unity, while in (1) every parameter is augmented.

Further we may deduce from (1)–(3) the differential equation satisfied by F, namely

$$\delta \prod_s (\delta+c_s-1)F = x \prod_r (\delta+a_r)F. \tag{4}$$

If now in (1)–(3) we set $x = 1$, the operation of differentiation becomes meaningless but δ, $\delta+a_r$, $\delta+c_s-1$, may now be regarded as

* Used in another connexion by T. W. Chaundy, a slightly different proof of Saalschütz's theorem being implicit in his analysis, (**2**) 68–9.

† (**1**) 113.

operating on the parameters a_r, c_s, with effects defined by the above identities. It is, however, important to observe that, whereas, when $|x| < 1$, the series F may be differentiated as often as we please, the series $F(a_r; c_s; 1)$ only converges if the real part of its 'index of convergence' $\sum c_s - \sum a_r$ is greater than zero, and that this index is diminished by unity by each δ operation of the type considered. If no initial restriction is placed upon the values of a_r, c_s, we may provisionally assume that their values are such as to make all series considered convergent and we may then, from (1)–(3) with $x = 1$, ascend to

$$\delta \prod (\delta + c_s - 1)F = \prod (\delta + a_r)F. \tag{5}$$

Any theorem deduced from this relation will be proved for a restricted range of parameter values, and, if the theorem is significant for a wider range of values, the restriction may then be removed by an appeal to the well-worn principle of analytic continuation.

If, however, as in Dougall's theorem, there is a pre-assigned relation between the parameters, that relation may be inconsistent with the convergence of some or all of the series concerned; the definitions (1)–(3) may fail and, even if they remain valid, the number of operations necessary may render the deduction of (5) impossible. In these circumstances results may still be deduced for terminating series, and we seem to have here a clue to the very different behaviour of the terminating and non-terminating types. At present we are concerned with the former only.

3. Saalschütz's Theorem

Let
$$e + f = a + b + c + 2 \tag{6}$$
and let c be a negative integer $-n$.

Then ${}_3F_2\left[\begin{matrix} a, b, c; \\ e, f \end{matrix}\right]$ satisfies the relation

$$[\delta(\delta + e - 1)(\delta + f - 1) - (\delta + a)(\delta + b)(\delta + c)]F = 0. \tag{7}$$

In virtue of (6) the bracketed operator is linear in δ and may be written in the form
$$A(\delta + a) - C(\delta + c),$$
where, setting in turn $\delta = -c$ and $\delta = -a$, we obtain

$$A(c - a) = c(c + 1 - e)(c + 1 - f), \qquad C(c - a) = a(a + 1 - e)(a + 1 - f).$$

Thus, from (2),

$$(a + 1 - e)(a + 1 - f)\,{}_3F_2\left[\begin{matrix} a, b, c + 1; \\ e, f \end{matrix}\right] = (c + 1 - e)(c + 1 - f)\,{}_3F_2\left[\begin{matrix} a + 1, b, c; \\ e, f \end{matrix}\right], \tag{8}$$

If in (8) we change a into $a-1$ and replace c by $-n$, we have in place of (6) the Saalschützian relation

$$e+f = a+b+1-n, \tag{9}$$

and, from (8),

$${}_3F_2\left[\begin{matrix}a, b, -n;\\ e, f\end{matrix}\right] = \frac{(e-a)(f-a)}{(e+n-1)(f+n-1)}\,{}_3F_2\left[\begin{matrix}a-1, b, -n+1;\\ e, f\end{matrix}\right].$$

The parameters of the ${}_3F_2$ on the right still satisfy (9), and the process may be repeated, the integral parameter n being reduced by unity at each stage. Thus eventually we have, subject to (9),

$${}_3F_2\left[\begin{matrix}a, b, -n;\\ e, f\end{matrix}\right] = \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n},$$

which is Saalschütz's theorem.

4. Dougall's Theorem

Let
$$a+b+c+d+e+f = 0 \tag{10}$$

and
$$a+f = -m \text{ (a negative integer).}$$

Also let

$$z = x^a\,{}_6F_5\left[\begin{matrix}2a, a+b, a+c, a+d, a+e, a+f\\ 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f\end{matrix}; x\right],$$

so that

$$\delta z = ax^a\,{}_7F_6\left[\begin{matrix}2a, a+1, a+b, a+c, a+d, a+e, a+f\\ a, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f\end{matrix}; x\right]$$
$$= a\phi(a,e,x).$$

Then, by (1) and (2),

$$(\delta^2-a^2)\phi = \frac{2(2a+1)(a+1)(a+b)(a+c)(a+d)(a+e)(a+f)}{(1+a-b)(1+a-c)(1+a-d)(1+a-e)(1+a-f)}\phi(a+1,e,x), \tag{11}$$

while, by (2) and (3),

$$(\delta^2-e^2)\phi = (a+e)(a-e)\phi(a,e+1,x). \tag{12}$$

Now z satisfies the differential equation

$$x\prod(\delta+a)z = \prod(\delta-a)z,$$

where the product is taken over the letters a, b, c, d, e, f, and, in virtue of (10),

$$\prod(\delta+a)-\prod(\delta-a) \equiv (A\delta^2+B)\delta,$$

where A and B are certain constants.

We may thus write

$$\prod(\delta+a)-\prod(\delta-a) \equiv \{P(\delta^2-a^2)-Q(\delta^2-e^2)\}2\delta$$

where, on setting in turn $\delta = e, a$, we obtain

$$P(e^2-a^2) = (e+a)(e+b)(e+c)(e+d)(e+f),$$
$$Q(e^2-a^2) = (a+b)(a+c)(a+d)(a+e)(a+f).$$

Since $$P(\delta^2-a^2)\phi(a,e,1) = Q(\delta^2-e^2)\phi(a,e,1)$$

we have, using (11) and (12) with $x = 1$,

$$\frac{\phi(a,e+1,1)}{\phi(a+1,e,1)} = \frac{(2a+1)(2a+2)(e+b)(e+c)(e+d)(e+f)}{(a-e)(1+a-e)(1+a-b)(1+a-c)(1+a-d)(1+a-f)},$$

i.e., employing (10) and setting $a+f = -m$,

$$\frac{\phi(a,e+1,1)}{\phi(a+1,e,1)} = \frac{(2a+1)(2a+2)(c+d-m)(b+d-m)(b+c-m)(e-a-m)}{(a-e)(1+a-e)(1+a-b)(1+a-c)(1+a-d)(1+2a+m)}. \tag{13}$$

Since the arguments of the two ϕ-functions of (13) are interchanged by augmenting a and diminishing e by unity, the relation (10) is undisturbed and the process may be repeated on $\phi(a+1,e,1)$. Also, since $a+f+m = 0$,

$$\phi(a+m,e+1-m,1) = 1,$$

and so

$$\phi(a,e+1,1)$$
$$= \frac{(2a+1)_{2m}(1-c-d)_m(1-b-d)_m(1-b-c)_m(a-e+m)_m}{(-)^m(a-e)_{2m}(1+a-b)_m(1+a-c)_m(1+a-d)_m(1+2a+m)_m}$$
$$= \frac{(2a+1)_m(1-c-d)_m(1-b-d)_m(1-b-c)_m}{(1-a-b-c-d)_m(1+a-b)_m(1+a-c)_m(1+a-d)_m}.$$

Thus, finally, changing e into $e-1$, we have Dougall's theorem which with the parameters here employed may be stated thus:

If $b+c+d+e = m+1$, *where* m *is a positive integer, then*

$${}_7F_6\left[\begin{matrix}2a, a+1, a+b, a+c, a+d, a+e, -m;\\ a, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+2a+m\end{matrix}\right]$$
$$= \frac{(2a+1)_m(1-c-d)_m(1-b-d)_m(1-b-c)_m}{(1-a-b-c-d)_m(1+a-b)_m(1+a-c)_m(1+a-d)_m}. \tag{14}$$

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1. W. N. Bailey, 'A new proof of Dixon's theorem', *Quart. J. of Math.* (Oxford) 8 (1937), 113.
2. T. W. Chaundy, 'An extension of hypergeometric functions', ibid. 14 (1943), 55–78.

SOME EIGENFUNCTION FORMULAE

By D. B. SEARS and E. C. TITCHMARSH (*Oxford*)

[Received 22 April 1949]

1. THE object of this paper is to correct § 4.14 and § 5.8 of *Eigenfunction Expansions associated with Second-order Differential Equations* by E. C. Titchmarsh. The example there given is inconsistent with the general theorem, and actually both of them are incorrect.

It is a question of the spectrum associated with the differential equation

$$\frac{d^2\phi}{dx^2}+\{\lambda-q(x)\}\phi = 0 \quad (0 \leqslant x < \infty) \tag{1.1}$$

in the case where $q'(x) < 0$, $q(x) \to -\infty$,

$$q'(x) = O\{|q(x)|^c\} \quad (0 < c < \tfrac{3}{2}),$$

$q''(x)$ is ultimately of one sign, and

$$\int^{\infty} |q(x)|^{-\frac{1}{2}}\, dx \tag{1.2}$$

is convergent. (The condition $q(x) \leqslant 0$ imposed is not really relevant.) The analysis given in § 5.8 is correct so far as positive values of λ are concerned, and the result stated, that the spectrum is discrete for $\lambda > 0$, is correct. *The actual result, however, is that under the above conditions the whole spectrum is discrete* (and not that it is continuous in $(-\infty, 0)$, as is stated in the book). The mistake arises in the first formula on p. 109. This contains the function

$$\xi(t) = \xi(t, \lambda) = \int_0^t \{\lambda-q(x)\}^{\frac{1}{2}}\, dx,$$

which may have a branch-point at $\lambda = q(0)$. The conclusion that $\mu(\lambda)$ is an integral function is therefore false, and the argument based on it fails.

The result will be proved by an appropriate modification of the method used in the book, although the proof may be shortened by appealing to a theorem of Weyl,† which is to the effect that the spectrum associated with (1.1) and certain boundary conditions is discrete when the equation is of limit-circle type.

The main mistake in § 4.14 is that the discrete spectrum arising from the zeros of $\sin(i\pi\sqrt{\lambda})$ is ignored.

† H. Weyl, *Math. Annalen*, 68 (1910), 220–69, § 8; also E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 12 (1941), 33–50, § 9.

2. Let $\phi(x) = \phi(x, \lambda)$ be the solution of (1.1) which satisfies the boundary conditions

$$\phi(0) = \sin\alpha, \qquad \phi'(0) = -\cos\alpha, \tag{2.1}$$

where α is a given real number. Then $\phi(x, \lambda)$ is an integral function of λ, for each x.

As in the book, let

$$\xi(x, \lambda) = \int_0^x \{\lambda - q(t)\}^{\frac{1}{2}}\, dt,$$

let
$$R(x, \lambda) = -\frac{q''(x)}{4\{\lambda - q(x)\}^{\frac{3}{2}}} - \frac{5}{16}\frac{q'^2(x)}{\{\lambda - q(x)\}^{\frac{5}{2}}},$$

and let
$$p(x, \lambda) = \{\lambda - q(x)\}^{-\frac{1}{4}}.$$

Let ρ be a real constant chosen so that $\rho - q(x) > 0$ for $0 \leqslant x < \infty$. Then

$$\begin{aligned}
&\int_0^x p(t,\rho)\{q(t)-\lambda\}\phi(t)\sin\{\xi(x,\rho)-\xi(t,\rho)\}\, dt \\
&\quad = \int_0^x p(t,\rho)\phi''(t)\sin\{\xi(x,\rho)-\xi(t,\rho)\}\, dt \\
&\quad = -p(0,\rho)\phi'(0)\sin\xi(x,\rho) - \int_0^x \phi'(t)[p'(t,\rho)\sin\{\xi(x,\rho)-\xi(t,\rho)\} - \\
&\qquad\qquad -p(t,\rho)\{\rho-q(t)\}^{\frac{1}{2}}\cos\{\xi(x,\rho)-\xi(t,\rho)\}]\, dt \\
&\quad = p(0,\rho)\cos\alpha\sin\xi(x,\rho) + \phi(x)\{\rho-q(x)\}^{\frac{1}{4}} + \\
&\qquad + \sin\alpha\, p'(0,\rho)\sin\xi(x,\rho) - \sin\alpha\{\rho-q(0)\}^{\frac{1}{4}}\cos\xi(x,\rho) + \\
&\qquad\quad + \int_0^x \phi(t)[p''(t,\rho) - p(t,\rho)\{\rho-q(t)\}]\sin\{\xi(x,\rho)-\xi(t,\rho)\}\, dt -
\end{aligned}$$

$$- \int_0^x \phi(t)\left[2p'(t,\rho)\{\rho-q(t)\}^{\frac{1}{2}} - \tfrac{1}{2}p(t,\rho)\frac{q'(t)}{\{\rho-q(t)\}^{\frac{1}{2}}}\right]\cos\{\xi(x,\rho)-\xi(t,\rho)\}\, dt.$$

Hence

$$\begin{aligned}
\phi(x)\{\rho-q(x)\}^{\frac{1}{4}} &= -\left\{\frac{\cos\alpha}{\{\rho-q(0)\}^{\frac{1}{4}}} + \frac{q'(0)\sin\alpha}{4\{\rho-q(0)\}^{\frac{5}{4}}}\right\}\sin\xi(x,\rho) + \\
&\quad + \sin\alpha\{\rho-q(0)\}^{\frac{1}{4}}\cos\xi(x,\rho) + \\
&\quad + \int_0^x \phi(t)\{\rho-q(t)\}^{\frac{1}{4}}\left\{\frac{\rho-\lambda}{\{\rho-q(t)\}^{\frac{1}{2}}} + R(t,\rho)\right\}\sin\{\xi(x,\rho)-\xi(t,\rho)\}\, dt.
\end{aligned} \tag{2.2}$$

If $\rho = \lambda$, this reduces to formula (5.4.5) of the book.

Writing
$$\phi_1(x) = \phi(x)\{\rho - q(x)\}^{\frac{1}{4}},$$
$$R_1(t) = \frac{\rho - \lambda}{\{\rho - q(t)\}^{\frac{1}{2}}} + R(t, \rho),$$
we have
$$\int_0^\infty |R_1(t)|\, dt < \infty.$$

Then, as in § 5.5, (2.2) is of the form
$$\phi_1(x) = A\cos\xi(x,\rho) + B\sin\xi(x,\rho) + \int_0^x \phi_1(t) R_1(t) \sin\{\xi(x,\rho) - \xi(t,\rho)\}\, dt. \quad (2.3)$$

It follows, as in § 5.6, firstly that $\phi_1(x)$ is bounded (uniformly in any finite λ-region), and secondly that, as $x \to \infty$,
$$\phi_1(x) = A\cos\xi(x,\rho) + B\sin\xi(x,\rho) + \int_0^\infty \phi_1(t) R_1(t) \sin\{\xi(x,\rho) - \xi(t,\rho)\}\, dt + o(1),$$
where A and B are independent of λ, and the integral converges uniformly over any finite λ-region, and so represents an integral function of λ. We have, therefore, as $x \to \infty$,
$$\phi(x)\{\rho - q(x)\}^{\frac{1}{4}} = \gamma(\lambda)\cos\xi(x,\rho) + \delta(\lambda)\sin\xi(x,\rho) + o(1), \quad (2.4)$$
where $\gamma(\lambda)$ and $\delta(\lambda)$ are integral functions of λ. (The difference between this formula and formula (5.7.3) of the book lies, of course, in the fact that here ρ is fixed, whereas in (5.7.3) $\xi(x) = \xi(x, \lambda)$ with varying λ.)

Similarly by using the differentiated form of (2.3) we obtain
$$\phi'(x)\{\rho - q(x)\}^{-\frac{1}{4}} = \delta(\lambda)\cos\xi(x,\rho) - \gamma(\lambda)\sin\xi(x,\rho) + o(1). \quad (2.5)$$

Similarly, if $\theta(x, \lambda)$ is the solution of (1.1) such that
$$\theta(0,\lambda) = \cos\alpha, \qquad \theta'(0,\lambda) = \sin\alpha,$$
we have
$$\theta(x)\{\rho - q(x)\}^{\frac{1}{4}} = \gamma_1(\lambda)\cos\xi(x,\rho) + \delta_1(\lambda)\sin\xi(x,\rho) + o(1), \quad (2.6)$$
$$\theta'(x)\{\rho - q(x)\}^{-\frac{1}{4}} = \delta_1(\lambda)\cos\xi(x,\rho) - \gamma_1(\lambda)\sin\xi(x,\rho) + o(1), \quad (2.7)$$
where $\gamma_1(\lambda)$, $\delta_1(\lambda)$ are obtained from $\gamma(\lambda)$ and $\delta(\lambda)$ by replacing $\sin\alpha$, $-\cos\alpha$, ϕ by $\cos\alpha$, $\sin\alpha$, θ respectively. Thus $\gamma_1(\lambda)$ and $\delta_1(\lambda)$ are also integral functions of λ.

We note that $\theta(x, \lambda)$ and $\phi(x, \lambda)$ are both $L^2(0, \infty)$ for all values of λ,

so that we are in Weyl's limit-circle case. According to the general theory, we have to consider the limit of

$$l(\lambda) = -\frac{\theta(b,\lambda)\cot\beta+\theta'(b,\lambda)}{\phi(b,\lambda)\cot\beta+\phi'(b,\lambda)}$$

as $b \to \infty$.

Substituting (2.4)–(2.7) on the right-hand side, and putting

$$\cot\beta = \{\rho-q(b)\}^{\frac{1}{2}}\cot\beta',$$

we obtain

$$l(\lambda) = -\frac{\gamma_1(\lambda)\cos\{\xi(b,\rho)+\beta'\}+\delta_1(\lambda)\sin\{\xi(b,\rho)+\beta'\}+o(1)}{\gamma(\lambda)\cos\{\xi(b,\rho)+\beta'\}+\delta(\lambda)\sin\{\xi(b,\rho)+\beta'\}+o(1)}.$$

Choosing β' as a function of b so that $\xi(b,\rho)+\beta' = \kappa$, we obtain

$$l(\lambda) \to -\frac{\gamma_1(\lambda)\cos\kappa+\delta_1(\lambda)\sin\kappa}{\gamma(\lambda)\cos\kappa+\delta(\lambda)\sin\kappa}.$$

As κ varies this describes a circle, which is therefore the limit-circle. Thus, in the notation of the book,

$$m(\lambda) = -\frac{\gamma_1(\lambda)\cos\kappa+\delta_1(\lambda)\sin\kappa}{\gamma(\lambda)\cos\kappa+\delta(\lambda)\sin\kappa}.$$

For any κ this is a meromorphic function of λ. The eigenvalues are its poles, and so the whole spectrum is discrete.

3. As an example, we shall consider the case $q(x) = -e^{2x}$, with $\alpha = 0$ in the boundary conditions at $x = 0$. Solutions of the equation

$$\frac{d^2\phi}{dx^2}+(\lambda+e^{2x})\phi = 0 \tag{3.1}$$

are $J_\nu(e^x)$, $J_{-\nu}(e^x)$, where $\nu = i\sqrt{\lambda}$. Since

$$J_\nu(z)J'_{-\nu}(z)-J'_\nu(z)J_{-\nu}(z) = -\frac{2\sin\nu\pi}{\pi z}, \tag{3.2}$$

we have

$$\theta(x,\lambda) = -\frac{\pi}{2\sin\nu\pi}\{J_\nu(e^x)J'_{-\nu}(1)-J_{-\nu}(e^x)J'_\nu(1)\}, \tag{3.3}$$

$$\phi(x,\lambda) = -\frac{\pi}{2\sin\nu\pi}\{J_\nu(e^x)J_{-\nu}(1)-J_{-\nu}(e^x)J_\nu(1)\}. \tag{3.4}$$

We can take $\rho = 0$, so that

$$\xi(x,\rho) = \xi(x,0) = \int_0^x e^t\,dt = e^x-1.$$

Thus the general theory gives

$$e^{\frac{1}{2}x}\phi(x,\lambda) = \gamma(\lambda)\cos(e^x-1)+\delta(\lambda)\sin(e^x-1)+o(1),$$

$$e^{\frac{1}{2}x}\theta(x,\lambda) = \gamma_1(\lambda)\cos(e^x-1)+\delta_1(\lambda)\sin(e^x-1)+o(1).$$

In fact it is known that

$$J_\nu(z) \sim \sqrt{\left(\frac{2}{\pi z}\right)}\cos(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi).$$

Hence

$$\phi(x) \sim -\frac{\sqrt{\pi}}{\sqrt{2}\sin\nu\pi}\, e^{-\frac{1}{2}x}\{J_{-\nu}(1)\cos(e^x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)-J_\nu(1)\cos(e^x+\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\}.$$

Comparing the two formulae, we obtain

$$\gamma(\lambda) = -\frac{\sqrt{\pi}}{\sqrt{2}\sin\nu\pi}\{J_{-\nu}(1)\cos(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1)-J_\nu(1)\cos(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1)\}, \quad (3.5)$$

$$\delta(\lambda) = -\frac{\sqrt{\pi}}{\sqrt{2}\sin\nu\pi}\{J_{-\nu}(1)\sin(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1)+J_\nu(1)\sin(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1)\}, \quad (3.6)$$

and similarly for $\gamma_1(\lambda)$ and $\delta_1(\lambda)$ with $J'_\nu(1)$, $J'_{-\nu}(1)$ replacing $J_\nu(1)$, $J_{-\nu}(1)$. Hence the denominator in $m(\lambda)$ is

$$\begin{aligned}&\{J_{-\nu}(1)\cos(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1)-J_\nu(1)\cos(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1)\}\cos\kappa+\\ &\quad+\{J_{-\nu}(1)\sin(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1)+J_\nu(1)\sin(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1)\}\sin\kappa\\ &\quad\quad= J_{-\nu}(1)\cos(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1-\kappa)-J_\nu(1)\cos(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1+\kappa),\end{aligned}$$

and similarly for the numerator. Thus

$$m(\lambda) = -\frac{J'_{-\nu}(1)\cos(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1-\kappa)-J'_\nu(1)\cos(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1+\kappa)}{J_{-\nu}(1)\cos(\tfrac{1}{2}\nu\pi+\tfrac{1}{4}\pi-1-\kappa)-J_\nu(1)\cos(\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi+1+\kappa)}. \quad (3.7)$$

Taking e.g. $1+\kappa = \frac{1}{4}\pi$, we obtain

$$m(\lambda) = -\frac{J'_{-\nu}(1)-J'_\nu(1)}{J_{-\nu}(1)-J_\nu(1)}. \quad (3.8)$$

The corresponding function

$$\psi(x,\lambda) = \theta(x,\lambda)+m(\lambda)\phi(x,\lambda)$$

is

$$\psi(x,\lambda) = \frac{J_\nu(e^x)-J_{-\nu}(e^x)}{J_\nu(1)-J_{-\nu}(1)} \quad (3.9)$$

and the expansion formula is easily obtained. Similarly, if $1+\kappa = \frac{3}{4}\pi$, then

$$m(\lambda) = -\frac{J'_\nu(1)+J'_{-\nu}(1)}{J_\nu(1)+J_{-\nu}(1)} \quad (3.10)$$

and

$$\psi(x,\lambda) = \frac{J_\nu(e^x)+J_{-\nu}(e^x)}{J_\nu(1)+J_{-\nu}(1)}. \quad (3.11)$$

4. Consider next the case where $q(x) = -e^{2x}$, and where the interval is $-\infty < x < \infty$. If we were considering the interval $-\infty < x < 0$ only, we should obtain a continuous spectrum in $\lambda > 0$, by the theory of § 5.3, since $|q(x)|$ is integrable over $(-\infty, 0)$. It follows that, in the case of the

whole interval $-\infty < x < \infty$, the spectrum will be discrete in $\lambda < 0$ and continuous in $\lambda > 0$. This we shall now verify directly.

Let $\theta(x,\lambda)$, $\phi(x,\lambda)$ denote the same functions as in § 3, and let

$$\psi_1(x,\lambda) = \theta(x,\lambda)+m_1(\lambda)\phi(x,\lambda),$$
$$\psi_2(x,\lambda) = \theta(x,\lambda)+m_2(\lambda)\phi(x,\lambda)$$

be solutions of (3.1) which are $L^2(-\infty,0)$ and $L^2(0,\infty)$ respectively. Now as $x \to -\infty$, $e^x \to 0$, and

$$J_\nu(e^x) \sim \frac{(\frac{1}{2}e^x)^\nu}{\Gamma(\nu+1)} = \frac{e^{ix\sqrt{\lambda}}}{2^\nu\Gamma(\nu+1)}.$$

If $\sqrt{\lambda} = \sigma+it$ ($\sigma > 0$, $t > 0$), then $|e^{ix\sqrt{\lambda}}| = e^{-xt}$, so that $J_\nu(e^x)$ is not $L^2(-\infty,0)$; but $J_{-\nu}(e^x)$ is $L^2(-\infty,0)$. Hence we must have

$$m_1(\lambda) = -\frac{J'_{-\nu}(1)}{J_{-\nu}(1)}, \tag{4.1}$$

and so

$$\psi_1(x,\lambda) = \frac{J_{-\nu}(e^x)}{J_{-\nu}(1)}. \tag{4.2}$$

The interval $(0,\infty)$ gives the limit-circle case as in § 3, and so involves an arbitrary parameter. We shall consider the two particular cases referred to in § 3. We may first take $m_2(\lambda)$ to be defined by (3.8), and $\psi_2(x,\lambda)$ by (3.9). The expansion formula will then be obtained by integrating with respect to λ the function

$$\Phi(x,\lambda) = \frac{\psi_2(x,\lambda)}{m_1(\lambda)-m_2(\lambda)}\int_{-\infty}^{x}\psi_1(\xi,\lambda)f(\xi)\,d\xi+$$
$$+\frac{\psi_1(x,\lambda)}{m_1(\lambda)-m_2(\lambda)}\int_{x}^{\infty}\psi_2(\xi,\lambda)f(\xi)\,d\xi$$
$$=\frac{\pi}{2\sin\nu\pi}\{J_\nu(e^x)-J_{-\nu}(e^x)\}\int_{-\infty}^{x}J_{-\nu}(e^\xi)f(\xi)\,d\xi+$$
$$+\frac{\pi}{2\sin\nu\pi}J_{-\nu}(e^x)\int_{x}^{\infty}\{J_\nu(e^\xi)-J_{-\nu}(e^\xi)\}f(\xi)\,d\xi. \tag{4.3}$$

This has a branch-point at $\lambda = 0$. The factor $\sin\nu\pi = \sin(i\sqrt{\lambda}\pi)$ vanishes at the points $\lambda = -n^2$ ($n = 1,2,\ldots$), but owing to the relation

$$J_{-n}(z) = (-1)^n J_n(z)$$

there is a pole only if n is odd, say for $n = 2m+1$ ($m = 0,1,\ldots$). The residue is

$$(4m+2)J_{2m+1}(e^x)\int_{-\infty}^{\infty}J_{2m+1}(e^\xi)f(\xi)\,d\xi.$$

Allowing for these residues, the integral

$$\frac{1}{2\pi i}\int \Phi(x,\lambda)\,d\lambda,$$

taken round a large circle in the positive direction, may be reduced to an integral round a loop starting at $+\infty$, encircling the origin in the positive direction, and returning to $+\infty$. The contribution to this of the first term in (4.3) is

$$\begin{aligned}
&-\frac{1}{4i}\int_0^\infty \frac{J_\nu(e^x)-J_{-\nu}(e^x)}{\sin\nu\pi}\,d\lambda \int_{-\infty}^{x} J_{-\nu}(e^\xi)f(\xi)\,d\xi+\\
&\qquad\qquad +\frac{1}{4i}\int_0^\infty \frac{J_{-\nu}(e^x)-J_\nu(e^x)}{\sin(-\nu\pi)}\,d\lambda \int_{-\infty}^{x} J_\nu(e^\xi)f(\xi)\,d\xi\\
&\quad=\frac{1}{4i}\int_0^\infty \frac{J_\nu(e^x)-J_{-\nu}(e^x)}{\sin\nu\pi}\,d\lambda \int_{-\infty}^{x} [J_\nu(e^\xi)-J_{-\nu}(e^\xi)]f(\xi)\,d\xi\\
&\quad=-\frac{1}{4}\int_0^\infty \frac{J_{i\sqrt\lambda}(e^x)-J_{-i\sqrt\lambda}(e^x)}{\sinh\pi\sqrt\lambda}\,d\lambda \int_{-\infty}^{x} [J_{i\sqrt\lambda}(e^\xi)-J_{-i\sqrt\lambda}(e^\xi)]f(\xi)\,d\xi,
\end{aligned}$$

and similarly for the other part.

Hence the complete expansion formula is

$$\begin{aligned}
f(x)=&\sum_{m=0}^{\infty}(4m+2)J_{2m+1}(e^x)\int_{-\infty}^{\infty}J_{2m+1}(e^\xi)f(\xi)\,d\xi-\\
&-\int_0^\infty \frac{J_{i\sqrt\lambda}(e^x)-J_{-i\sqrt\lambda}(e^x)}{4\sinh\pi\sqrt\lambda}\,d\lambda\int_{-\infty}^{\infty}[J_{i\sqrt\lambda}(e^\xi)-J_{-i\sqrt\lambda}(e^\xi)]f(\xi)\,d\xi. \qquad (4.4)
\end{aligned}$$

Similarly, if $m_2(\lambda)$ is defined by (3.10), and $\psi_2(x,\lambda)$ by (3.11), we obtain

$$\begin{aligned}
\Phi(x,\lambda)=&\frac{\pi}{2\sin\nu\pi}\{J_\nu(e^x)+J_{-\nu}(e^x)\}\int_{-\infty}^{x}J_{-\nu}(e^\xi)f(\xi)\,d\xi+\\
&+\frac{\pi}{2\sin\nu\pi}J_{-\nu}(e^x)\int_x^\infty\{J_\nu(e^\xi)+J_{-\nu}(e^\xi)\}f(\xi)\,d\xi,
\end{aligned}$$

and the expansion formula is

$$\begin{aligned}
f(x)=&\sum_{m=1}^{\infty}4mJ_{2m}(e^x)\int_{-\infty}^{\infty}J_{2m}(e^\xi)f(\xi)\,d\xi+\\
&+\int_0^\infty\frac{J_{i\sqrt\lambda}(e^x)+J_{-i\sqrt\lambda}(e^x)}{4\sinh\pi\sqrt\lambda}\,d\lambda\int_{-\infty}^{\infty}[J_{i\sqrt\lambda}(e^\xi)+J_{-i\sqrt\lambda}(e^\xi)]f(\xi)\,d\xi. \qquad (4.5)
\end{aligned}$$

If we put $e^x = s$, $e^\xi = t$, and

$$f(x) = f(\log s) = g(s),$$

these formulae take the form

$$g(s) = \sum_{m=0}^{\infty} (4m+2) J_{2m+1}(s) \int_0^\infty J_{2m+1}(t) g(t) \frac{dt}{t} -$$

$$- \int_0^\infty \frac{J_{i\sqrt{\lambda}}(s) - J_{-i\sqrt{\lambda}}(s)}{4 \sinh \pi\sqrt{\lambda}} d\lambda \int_0^\infty [J_{i\sqrt{\lambda}}(t) - J_{-i\sqrt{\lambda}}(t)] g(t) \frac{dt}{t} \quad (4.6)$$

and

$$g(s) = \sum_{m=1}^{\infty} 4m J_{2m}(s) \int_0^\infty J_{2m}(t) g(t) \frac{dt}{t} +$$

$$+ \int_0^\infty \frac{J_{i\sqrt{\lambda}}(s) + J_{-i\sqrt{\lambda}}(s)}{4 \sinh \pi\sqrt{\lambda}} d\lambda \int_0^\infty [J_{i\sqrt{\lambda}}(t) + J_{-i\sqrt{\lambda}}(t)] g(t) \frac{dt}{t}. \quad (4.7)$$

5. We shall discuss now the relation between the above formulae and the Webb–Kapteyn theory of Neumann series.†

A Neumann series for an odd function $g(s)$ is of the form

$$\sum_{m=0}^{\infty} a_{2m+1} J_{2m+1}(s), \quad (5.1)$$

and it is known that certain classes of analytic functions can be expanded in this form. However, in view of the formula

$$\int_0^\infty J_{2m+1}(t) J_{2n+1}(t) \frac{dt}{t} = \begin{cases} 0 & (m \neq n), \\ 1/(4n+2) & (m = n) \end{cases} \quad (5.2)$$

it is also possible to calculate the coefficients a_{2m+1} in the case of an arbitrary function of a real variable $g(s)$. Proceeding in the manner of an ordinary Fourier series, we obtain

$$a_{2m+1} = (4m+2) \int_0^\infty g(t) J_{2m+1}(t) \frac{dt}{t}. \quad (5.3)$$

It was shown, however, by Kapteyn, that the series (5.1), with the coefficients (5.3), can represent the function $g(s)$ only if $g(s)$ satisfies the integral equation

$$g'(s) = \frac{1}{2} \int_0^\infty \frac{J_1(t)}{t} \{g(t+s) + g(t-s)\}\, dt. \quad (5.4)$$

† G. N. Watson, *Theory of Bessel Functions*, § 16.4.

For an odd function $g(s)$, this is equivalent to

$$g'(s) = \frac{1}{2}\int_0^\infty \frac{J_1(t)}{t}\{g(s+t)-g(s-t)\}\,dt. \tag{5.5}$$

This equation was solved by Hardy and Titchmarsh.† It was shown, for example, that, if $g(s)$ is of integrable square over $(-\infty,\infty)$, then a necessary and sufficient condition for (5.5) to be satisfied is that $g(s)$ should be of the form

$$g(s) = \int_0^1 \{a(u)\cos su + b(u)\sin su\}\,du, \tag{5.6}$$

where $a(u)$ and $b(u)$ are $L^2(0,1)$. In particular it follows that $g(s)$ must be an integral function of s of a certain type.

The formulae of § 4 throw a new light on all these questions. The Webb–Kapteyn Neumann expansion of $g(s)$ is just the formula (4.6) without the repeated-integral term. The condition that it should be valid is that the continuous spectrum should make no contribution to the expansion. It is to be expected that this will hold only for special classes of functions.

To prove the sufficiency of the Hardy–Titchmarsh condition directly, suppose that $g(s)$ is of the form (5.6) and is odd, so that in fact it is of the form

$$g(s) = \int_0^1 b(u)\sin su\,du, \tag{5.7}$$

where $b(u)$ is $L^2(0,1)$. Then

$$\int_0^\infty \{J_{i\nu\lambda}(t)-J_{-i\nu\lambda}(t)\}g(t)\frac{dt}{t} = \int_0^1 b(u)\,du\int_0^\infty \{J_{i\nu\lambda}(t)-J_{-i\nu\lambda}(t)\}\frac{\sin tu}{t}\,dt, \tag{5.8}$$

the inversion being justified by absolute convergence. Now,‡ if $0 < u < 1$,

$$\int_0^\infty \frac{J_\nu(t)\sin tu}{t}\,dt = \frac{\sin(\nu \arcsin u)}{\nu}.$$

Hence the inner integral in (5.8) vanishes, and the result follows.

To prove a theorem of the converse type, we observe that the Parseval

† G. H. Hardy and E. C. Titchmarsh, *Proc. London Math. Soc.* (2), **23** (1923), 1–26.

‡ G. N. Watson, loc. cit. § 13.42 (2).

formula corresponding to (4.6) is

$$\int_0^\infty \frac{\{g(s)\}^2}{s}\,ds = \sum_{m=0}^{\infty}(4m+2)\left\{\int_0^\infty J_{2m+1}(t)g(t)\frac{dt}{t}\right\}^2 -$$

$$-\int_0^\infty \frac{1}{4\sinh \pi\sqrt{\lambda}}\left\{\int_0^\infty [J_{i\sqrt{\lambda}}(t)-J_{-i\sqrt{\lambda}}(t)]g(t)\frac{dt}{t}\right\}^2 d\lambda.$$

For the contribution of the continuous spectrum to vanish, the second term on the right must vanish, and therefore

$$\int_0^\infty [J_{i\sqrt{\lambda}}(t)-J_{-i\sqrt{\lambda}}(t)]\frac{g(t)}{t}\,dt = 0 \tag{5.9}$$

for all positive values of λ.

In order to avoid intricate analysis, we shall assume in this problem that $g(t)$ is absolutely continuous, that $g(t)$ and $g'(t)$ are both $L^2(0,\infty)$, and that $g(0) = 0$. As in Theorem 68 of Titchmarsh's *Fourier Integrals*, it follows that, if $\sqrt{(\frac{1}{2}\pi)}\,b(u)$ is the sine transform of $g(t)$, then $b(u)$ and $ub(u)$ are both $L^2(0,\infty)$.

Since $J_\nu(t)/t$ is $L^2(0,\infty)$ if $\mathbf{R}(\nu) > \frac{1}{2}$, we can calculate

$$\int_0^\infty \frac{J_\nu(t)}{t}g(t)\,dt$$

by the formulae of Fourier sine-transforms. We have

$$\int_0^\infty \frac{J_\nu(t)}{t}\sin ut\,dt = \begin{cases} \nu^{-1}\sin(\nu \arcsin u) & (u<1), \\ \dfrac{\sin\frac{1}{2}\nu\pi}{\nu[u+\sqrt{(u^2-1)}]^\nu} & (u>1). \end{cases}$$

Hence, if $\mathbf{R}(\nu) > \frac{1}{2}$,

$$\int_0^\infty \frac{J_\nu(t)}{t}g(t)\,dt = \int_0^1 \frac{\sin(\nu\arcsin u)}{\nu}b(u)\,du + \int_1^\infty \frac{\sin\frac{1}{2}\nu\pi}{\nu[u+\sqrt{(u^2-1)}]^\nu}b(u)\,du.$$

Under the above conditions, these integrals converge uniformly with respect to ν for $\mathbf{R}(\nu) > -\frac{1}{2}$, $|\nu| \geqslant \delta > 0$, so that the formula holds, by the theory of analytic continuation, in this region. In particular it holds if ν is purely imaginary. Taking $\nu = i\sqrt{\lambda}$, then $\nu = -i\sqrt{\lambda}$, and subtracting, we obtain, by (5.9),

$$\int_1^\infty \{[u+\sqrt{(u^2-1)}]^{i\sqrt{\lambda}} - [u+\sqrt{(u^2-1)}]^{-i\sqrt{\lambda}}\}b(u)\,du = 0;$$

or, writing $u = \cosh w$,

$$\int_0^\infty b(\cosh w)\sinh w \sin(w\sqrt{\lambda})\, dw = 0.$$

Since $$\int_0^\infty \{b(\cosh w)\sinh w\}^2\, dw = \int_1^\infty b^2(u)\sqrt{(u^2-1)}\, du < \infty$$

under the conditions assumed, it follows that

$$b(\cosh w)\sinh w = 0$$

for almost all values of w, i.e. that $b(u) = 0$ for almost all values of u greater than 1. Hence $g(s)$ is of the form (5.7).

ON THE INTEGRO-EXPONENTIAL FUNCTION AND THE EVALUATION OF SOME INTEGRALS INVOLVING IT

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1. Introduction

THE function $\int_1^\infty t^{-n}e^{-xt}\,dt$, where n is a positive integer and $x > 0$, has been variously denoted by† $E_n(x)$, $U_n(x)$, and $K_n(x)$. I shall adopt the notation $E_n(x)$ and write

$$E_n(x) = \int_1^\infty t^{-n}e^{-xt}\,dt.$$

The object of this paper is to extend the work of V. Kourganoff, whose two papers on the function and related integrals‡ will be referred to as K 1 and K 2.

The function $E_n(x)$ is becoming of increasing importance in problems connected with the radiative equilibrium of stellar atmospheres and in problems of neutron diffusion.§ In K 1 and K 2 Kourganoff has given an account of the properties of $E_n(x)$ and of its relation to other functions of analysis such as the incomplete gamma function, Whittaker functions, etc., and he has evaluated integrals of the form

$$I^1_{p,s,n}(a) = \int_0^\infty e^{-px}x^sE_n(ax)\,dx, \tag{1.1}$$

$$I^2_{p,s,n_1n_2}(a_1, a_2) = \int_0^\infty e^{-px}x^sE_{n_1}(a_1x)E_{n_2}(a_2x)\,dx, \tag{1.2}$$

where s, n, n_1, n_2 are integers ($s \geqslant 0$; n, n_1, $n_2 \geqslant 1$). These are of importance in the solutions of the physical problems referred to above.

† $E_n(x)$ has been used, for example, by E. Hopf in *Mathematical Problems of Radiative Equilibrium* (Cambridge Tract 31). $U_n(x)$ has been used by Eddington in *The Internal Constitution of the Stars* (Cambridge, 1926). $K_n(x)$ is the notation used by Kourganoff in K 1 and K 2 (see below).

‡ *Annales d'Astrophysique*, 10 (1947), 282–99 (K 1) and 329–40 (K 2).

§ See, for example, Kourganoff, *Comptes rendus*, 227 (1948), 1020–2.

It is of mathematical interest to extend the definition and properties of $E_n(x)$ to complex values of n and x, and I do this in § 2. In §§ 3–6 I consider integrals of the form

$$I^k_{p,\sigma,\nu_1,\ldots\nu_k}(a_1,\ldots,a_k) = \int_0^\infty e^{-px}x^\sigma E_{\nu_1}(a_1x)\ldots E_{\nu_k}(a_kx)\,dx, \tag{1.3}$$

where p, σ, $\nu_1,\ldots,\nu_k$, $a_1,\ldots,a_k$ are complex, and I deduce reduction formulae for the integrals (1.2) which have some advantages over those given in K 1 and K 2.

In the following paragraphs, σ, ν_i, p, a_i will denote complex numbers and s, n_i will denote positive integers (or zero).

2. The definition and properties of $E_\nu(z)$

When ν is any complex number and when $\mathbf{R}(z) > 0$, we define $E_\nu(z)$ by

$$E_\nu(z) = \int_1^\infty t^{-\nu}e^{-zt}\,dt, \tag{2.1}$$

where $t^\nu = \exp(\nu\log t)$. Then $E_\nu(z)$ is an analytic function of z which is regular for $\mathbf{R}(z) > 0$, and it is an integral function of ν. If $\mathbf{R}(\nu) > 1$, then

$$E_\nu(0) = (\nu-1)^{-1}.$$

If $\mathbf{R}(\nu) < 1$ and if z is real and positive,

$$E_\nu(z) = z^{\nu-1}\int_z^\infty u^{-\nu}e^{-u}\,du = z^{\nu-1}\left\{\Gamma(1-\nu) - \int_0^z e^{-u}u^{-\nu}\,du\right\},$$

and so we get

$$E_\nu(z) = z^{\nu-1}\Gamma(1-\nu) - \sum_{k=0}^\infty \frac{(-1)^k z^k}{k!(k-\nu+1)}, \tag{2.2}$$

a formula which provides the analytic continuation of $E_\nu(z)$ to the domain $|\arg z| < \pi$ except when ν is a positive integer. By letting $\nu \to n$ $(n \geqslant 1)$ in (2.2) we get the corresponding expression for $E_n(z)$, viz.

$$E_n(z) = \sum_{k=0}^{n-2}\frac{(-1)^{k-1}z^k}{k!\,(k-n+1)} + (-1)^n\frac{z^{n-1}}{(n-1)!}\left\{\log z + \gamma - \sum_{r=1}^{n-1}\frac{1}{r}\right\} + \\ + \sum_{k=n}^\infty \frac{(-1)^{k-1}z^k}{k!(k-n+1)} \qquad (|\arg z| < \pi;\ n \geqslant 1), \tag{2.3}$$

the first sum being absent when $n = 1$.

On writing $t = 1+u$ in (2.1), we get the formula

$$e^z E_\nu(z) = \int_0^\infty \frac{e^{-zu}}{(1+u)^\nu}\,du \quad (\mathbf{R}(z) > 0),$$

and by rotation of the line of integration it follows that

$$e^z E_\nu(z) = \int_0^\infty \frac{e^{-ze^{i\beta}}}{(1+te^{i\beta})^\nu} e^{i\beta}\, dt \quad (|\beta| < \tfrac{1}{2}\pi;\ \mathbf{R}(ze^{i\beta}) > 0). \tag{2.4}$$

By applying Watson's lemma† to (2.4), we at once obtain the asymptotic expansion of $E_\nu(z)$ for large $|z|$, viz.

$$E_\nu(z) \sim e^{-z}\left\{\frac{1}{z} - \frac{\nu}{z^2} + \frac{\nu(\nu+1)}{z^3} - \frac{\nu(\nu+1)(\nu+2)}{z^4} + \ldots\right\}. \tag{2.5}$$

By varying β in (2.4), this is found to hold for $|\arg z| \leqslant \pi - \delta$ $(\delta > 0)$.

Two other formulae are of great importance in the sequel. These are

$$E'_\nu(z) = -E_{\nu-1}(z), \tag{2.6}$$

$$(\nu-1)E_\nu(z) = e^{-z} - zE_{\nu-1}(z). \tag{2.7}$$

Both are easy deductions from (2.1) and both hold for $|\arg z| < \pi$.

3. The evaluation of $I^k_{p,\sigma,\nu_1\ldots\nu_k}$: general considerations

By (2.2) and (2.5), the integral

$$I^k_{p,\sigma,\nu_1\ldots\nu_k}(a_1,\ldots,a_k) = \int_0^\infty e^{-px} x^\sigma E_{\nu_1}(a_1 x)\ldots E_{\nu_k}(a_k x)\, dx, \tag{3.1}$$

where $|\arg a_i| < \pi$ for $i = 1, 2,\ldots, k$, will converge if

$$\mathbf{R}(p+a_1+\ldots+a_k) > 0$$

and if

$$\mathbf{R}(\sigma)+\mathbf{R}(\nu_{r_1}+\ldots+\nu_{r_p}-q+1) > 0, \tag{3.2}$$

where $\nu_{r_1},\ldots, \nu_{r_q}$ are those ν_i whose real part is less than unity. The condition (3.2) is always satisfied when $\sigma = s \geqslant 0$, $\nu_i = n_i \geqslant 1$ $(i = 1,\ldots, k)$.

[Because $E_0(px) = e^{-px}/px$, the integral could be written

$$p\int_0^\infty x^{\sigma+1} E_0(px) E_{\nu_1}(a_1 x)\ldots E_{\nu_k}(a_k x)\, dx$$

and this can be considered as a special case of $I^{k+1}_{0,\sigma,\nu_1\ldots\nu_{k+1}}$. No real simplification is gained by this and the fact that we are considering the Laplace transform of $x^\sigma E_{\nu_1}(a_1 x)\ldots E_{\nu_k}(a_k x)$ is lost.]

It is convenient to give a meaning to (3.1) when $k = 0$, so I define

$$I^0_{p,\sigma} = \int_0^\infty e^{-px} x^\sigma\, dx = \Gamma(\sigma+1)p^{-\sigma-1} \quad (\mathbf{R}(\sigma) > -1;\ \mathbf{R}(p) > 0). \tag{3.3}$$

† Copson, *Functions of a Complex Variable* (Oxford, 1935), 218.

I shall now show that relations (2.6) and (2.7) enable us to reduce $I^k_{p,\sigma,\nu_1,\dots,\nu_k}$ either

(i) *to* $I^k_{p,\sigma-1,\nu_1,\dots,\nu_k}$ *together with integrals of the form*† $I^{k-1}_{p,\sigma,\nu_1,\dots,\nu_{k-1}}$,

or (ii) *to* $I^k_{p,\sigma,\nu'_1,\dots,\nu'_k}$, *where each* ν'_i *may be either* ν_i *or* ν_i-1, *together with integrals of the form* $I^{k-r}_{p,\sigma,\nu_1,\dots,\nu_{k-r}}$.

From this it will follow that, provided that $I^{k-r}_{p,\sigma,\nu_1,\dots,\nu_{k-r}}$ can be evaluated for $1 \leqslant r \leqslant k$ and provided that an integral is known to which $I^k_{p,\sigma,\nu_1,\dots,\nu_k}$ can be reduced by reduction of $\sigma, \nu_1,\dots, \nu_k$ by integers, then $I^k_{p,\sigma,\nu_1,\dots,\nu_k}$ can be evaluated. In particular it will be possible to evaluate $I^k_{p,s,n_1,\dots,n_k}$ provided that integrals of the form $I^{k-r}_{p,s,n_1,\dots,n_{k-r}}$ $(1 \leqslant r \leqslant k)$ are known and provided that $I^k_{p,0,1,\dots,1}$ is known. This forms the basis of the evaluation of the integral (1.2).

Proof of (i). From (2.6) and (2.7) we have

$$zE'_\nu(z) = (\nu-1)E_\nu(z) - e^{-z}. \tag{3.4}$$

Using this we see that

$$\begin{aligned}\frac{d}{dx}\{e^{-px}x^\sigma E_{\nu_1}(a_1x)\dots E_{\nu_k}(a_kx)\} &= -pe^{-px}x^\sigma E_{\nu_1}(a_1x)\dots E_{\nu_k}(a_kx) + \\ &+(\sigma+\nu_1+\dots+\nu_k-k)e^{-px}x^{\sigma-1}E_{\nu_1}(a_1x)\dots E_{\nu_k}(a_kx) - \\ &-\sum_{i=1}^{k} e^{-(p+a_i)x}x^{\sigma-1}\prod_{j=1}{}' E_{\nu_j}(a_jx),\end{aligned} \tag{3.5}$$

where the prime denotes that the term $j = i$ is omitted. On integrating over $(0,\infty)$, the integrated terms vanish at the upper limit when $\mathbf{R}(p+a_1+\dots+a_k) > 0$, and we obtain

$$\begin{aligned}pI^k_{p,\sigma,\nu_1,\dots,\nu_k} &= (\sigma+\nu_1+\dots+\nu_k-k)I^k_{p,\sigma-1,\nu_1,\dots,\nu_k} - \\ &-\sum_{i=1}^{k} I^{k-1}_{p+a_i,\sigma-1,\nu_1,\dots,\nu_{i-1},\nu_{i+1},\dots,\nu_k} + \lim_{x\to 0} e^{-px}x^\sigma E_{\nu_1}(a_1x)\dots E_{\nu_k}(a_kx).\end{aligned} \tag{3.6}$$

This proves (i).

From (3.6) we see, in particular, on putting $p = 0$ and replacing σ by $\sigma+1$, that $I^k_{0,\sigma,\nu_1,\dots,\nu_k}$ can be evaluated when integrals of the form $I^{k-1}_{p,\sigma,\nu_1,\dots,\nu_{k-1}}$ are known. I shall return to this important case in § 5.

Proof of (ii). By (2.7) we can replace one or more of the $E_{\nu_i}(a_ix)$ $(\nu_i \neq 1)$ in $I^k_{p,\sigma,\nu_1,\dots,\nu_k}$ by

$$\frac{1}{\nu_i-1}\{e^{-a_ix} - a_ixE_{\nu_i-1}(a_ix)\}.$$

Let us suppose that $E_{\nu_1}(a_1x),\dots, E_{\nu_n}(a_nx)$ $(n \leqslant k)$ are so changed. On

† When I say that integrals are 'of the form $I^{k-r}_{p,\sigma,\nu_1,\dots,\nu_{k-r}}$', I do not mean that any of $p, \sigma, \nu_1,\dots, \nu_{k-r}$ are necessarily the same as those appearing in $I^k_{p,\sigma,\nu_1,\dots,\nu_k}$.

multiplying out we get integrals of the form $I^{k-r}_{p,\sigma,\nu_1,\dots,\nu_{k-r}}$ $(r=1,\dots,n)$ together with the integral $I^{k}_{p,\sigma+n,\nu_1-1,\dots,\nu_n-1,\nu_{n+1},\dots,\nu_k}$. But this can be reduced by (i) to $I^{k}_{p,\sigma,\nu_1-1,\dots,\nu_n-1,\nu_{n+1},\dots,\nu_k}$ and to integrals of the form $I^{k-1}_{p,\sigma,\nu_1,\dots,\nu_{k-1}}$, so (ii) is proved.

It will be noticed that, although integration over $(0,\infty)$ has been considered above, the same methods hold for integration over a finite interval (a,b) where $0 \leqslant a < b$.

4. The evaluation of $I^1_{p,\sigma,\nu}(a) = \int_0^\infty e^{-px}x^\sigma E_\nu(ax)\,dx$

$I^1_{p,s,n}(a)$ has been evaluated by Kourganoff in K 1. By a similar method, i.e. by substituting for $E_\nu(ax)$ from (2.1) and inverting the order of integration, we arrive at extensions of his results, viz.

THEOREM 1. *If* $\mathbf{R}(a)>0, \mathbf{R}(p+a)>0, \mathbf{R}(\sigma)+\min\{1,\mathbf{R}(\nu)\}>0$, *then*

$$I^1_{p,\sigma,\nu}(a) = \frac{\Gamma(\sigma+1)}{a^{\sigma+1}(\sigma+\nu)}F(\sigma+1,\sigma+\nu;\sigma+\nu+1;-p/a)$$

$$= \frac{\Gamma(\sigma+1)}{a(\sigma+\nu)(p+a)^\sigma}F(1,\nu;\sigma+\nu+1;-p/a). \tag{4.1}$$

The conditions $\mathbf{R}(a)>0, \mathbf{R}(p+a)>0$ together imply that $-p/a$ lies within the domain of existence of the hypergeometric function.

When $\sigma = s$, $\nu = n$, more useful results than those contained in Theorem 1 can be obtained by means of reduction formulae.† From (3.6) and (3.3) we get, since $\lim_{x\to 0} e^{-px}x^sE_n(ax) = 0$ when $s \geqslant 1$, $n \geqslant 1$,

FORMULA I. *If* $|\arg a| < \pi, \mathbf{R}(p+a)>0$ *and if* $s \geqslant 1$, $n \geqslant 1$, *then*

$$pI^1_{p,s,n}(a) - (n+s-1)I^1_{p,s-1,n}(a) = -(p+a)^{-s}\Gamma(s).$$

Using this, the reduction of $I^1_{p,s,n}$ to

$$I^1_{p,0,1}(a) = \frac{1}{p}\log\left(1+\frac{p}{a}\right) \tag{4.2}$$

can be completed by

FORMULA II. *If* $|\arg a| < \pi, \mathbf{R}(p+a)>0$ *and if* $n \geqslant 2$, *then*

$$pI^1_{p,0,n}(a) + aI^1_{p,0,n-1}(a) = \frac{1}{n-1}.$$

To prove this we use (2.6). Then

$$aI^1_{p,0,n-1}(a) = -a\int_0^\infty e^{-px}E'_n(ax)\,dx = E_n(0) - pI^1_{p,0,n}(a)$$

on integrating by parts. Since $E_n(0) = 1/(n-1)$, II follows.

† Kourganoff, in K 1, establishes a reduction formula connecting $I^1_{p,s,n}$ with $I^1_{p,s-1,n+1}$, thus reducing $I_{p,s,n}$ to $I_{p,0,n+s}$, which is easily evaluated (see (4.3)).

From II and (4.2) we easily obtain the important formula†

$$I^1_{p,0,n}(a) = \frac{(-1)^{n+1}}{a}\left(\frac{a}{p}\right)^n\left\{\log\left(1+\frac{p}{a}\right)+\phi_{n-1}\left(\frac{p}{a}\right)\right\}, \tag{4.3}$$

where
$$\phi_n(x) = \sum_{k=1}^{n}\frac{(-1)^k x^k}{k}, \qquad \phi_0(x) = 0. \tag{4.4}$$

Formulae I and II are true, also, when s, n are replaced by σ, ν provided that $\mathbf{R}(\sigma) > 0$ and $\mathbf{R}(\sigma+\nu) > 1$ in I and that $\mathbf{R}(\nu) > 1$ in II. If $\sigma+\nu = 1$, formula I breaks down, but we can get an interesting result if we return to (3.5) (with $k = 1$). This gives

$$pe^{-px}x^{1-\nu}E_\nu(ax) = -e^{-(p+a)x}x^{-\nu} - \frac{d}{dx}\{e^{-px}x^{1-\nu}E_\nu(ax)\}$$

and so, if $\mathbf{R}(\nu) < 1$,

$$\begin{aligned} pI^1_{p,1-\nu,\nu}(a) &= -\Gamma(1-\nu)(p+a)^{\nu-1} + \lim_{x\to 0} e^{-px}x^{1-\nu}E_\nu(ax) \qquad (4.5)\\ &= \Gamma(1-\nu)\{a^{\nu-1}-(p+a)^{\nu-1}\}, \end{aligned}$$

by (2.2).

5. The evaluation of $I^2_{0,\sigma,\nu_1\nu_2}(a_1, a_2) = \int_0^\infty x^\sigma E_{\nu_1}(a_1 x)E_{\nu_2}(a_2 x)\,dx$

From (3.6), with $p = 0$ and with $\sigma+1$ in place of σ, we have

$$\begin{aligned} &(\sigma+\nu_1+\nu_2-1)I^2_{0,\sigma,\nu_1\nu_2}(a_1, a_2)\\ &\quad = I^1_{a_1,\sigma,\nu_2}(a_2) + I^1_{a_2,\sigma,\nu_1}(a_1) - \lim_{x\to 0} x^{\sigma+1}E_{\nu_1}(a_1 x)E_{\nu_2}(a_2 x). \end{aligned}$$

But, for convergence of the integrals, we must have

$$\mathbf{R}(\sigma)+\min\{1, \mathbf{R}(\nu_1), \mathbf{R}(\nu_2), \mathbf{R}(\nu_1+\nu_2-1)\} > 0,$$

and so
$$\lim_{x\to 0} x^{\sigma+1}E_{\nu_1}(a_1 x)E_{\nu_2}(a_2 x) = 0.$$

Hence we get

THEOREM 2. *If* $|\arg a_1| < \pi$, $|\arg a_2| < \pi$, $\mathbf{R}(a_1+a_2) > 0$,

and $$\mathbf{R}(\sigma)+\min\{1, \mathbf{R}(\nu_1), \mathbf{R}(\nu_2), \mathbf{R}(\nu_1+\nu_2-1)\} > 0,$$

then
$$(\sigma+\nu_1+\nu_2-1)I^2_{0,\sigma,\nu_1\nu_2}(a_1, a_2) = I^1_{a_1,\sigma,\nu_2}(a_2) + I^1_{a_2,\sigma,\nu_1}(a_1). \tag{5.1}$$

If, in addition, $\mathbf{R}(a_1) > 0$, $\mathbf{R}(a_2) > 0$, *then*

$$\begin{aligned} &(\sigma+\nu_1+\nu_2-1)I^2_{0,\sigma,\nu_1\nu_2}(a_1, a_2)\\ &\quad = \frac{\Gamma(\sigma+1)}{a_2^{\sigma+1}(\sigma+\nu_2)}F\left(\sigma+1, \sigma+\nu_2;\ \sigma+\nu_2+1;\ -\frac{a_1}{a_2}\right) +\\ &\qquad + \frac{\Gamma(\sigma+1)}{a_1^{\sigma+1}(\sigma+\nu_1)}F\left(\sigma+1, \sigma+\nu_1; \sigma+\nu_1+1; -\frac{a_2}{a_1}\right). \qquad (5.2) \end{aligned}$$

† K 1 (6.8).

The special result given by $\sigma = 0$, $\nu_1 = n_1$, $\nu_2 = n_2$ is of importance. Using (4.3) we have†

COROLLARY. *If* $|\arg a_1| < \pi$, $|\arg a_2| < \pi$, $\mathbf{R}(a_1+a_2) > 0$, $n_1 \geqslant 1$, $n_2 \geqslant 1$, *then*

$$(n_1+n_2-1)I^2_{0,0,n_1n_2}(a_1,a_2) = \frac{(-1)^{n_1+1}}{a_1}\left(\frac{a_1}{a_2}\right)^{n_1}\left\{\log\left(1+\frac{a_2}{a_1}\right)+\phi_{n_1-1}\left(\frac{a_2}{a_1}\right)\right\}+$$

$$+\frac{(-1)^{n_2+1}}{a_2}\left(\frac{a_2}{a_1}\right)^{n_2}\left\{\log\left(1+\frac{a_1}{a_2}\right)+\phi_{n_2-1}\left(\frac{a_1}{a_2}\right)\right\}, \tag{5.3}$$

where $\phi_n(x)$ *is given by* (4.4).

If $a_1 = a_2 = 1$, it is usual to write $J_{m,n}$ for $I^2_{0,0,m,n}(1,1)$. With this notation, and using, also, a notation introduced by Kourganoff in K 1, we get the formula

$$(m+n-1)J_{m,n} = (m+n-1)\int_0^\infty E_m(x)E_n(x)\,dx$$

$$= \{(-1)^{m+1}+(-1)^{n+1}\}\log 2+a^1_m+a^1_n$$

where a^1_n is determined by

$$a^1_{n+1}+a^1_n = \frac{1}{n} \quad (n \geqslant 1), \qquad a^1_1 = 0.$$

One other form of (5.3) is of special interest. This is‡

$$I^2_{0,0,1,1}(a_1,a_2) = \frac{1}{a_1a_2}\log\frac{(a_1+a_2)^{a_1+a_2}}{a_1^{a_1}a_2^{a_2}}.$$

6. The evaluation of $I^2_{p,\sigma\,\nu_1\nu_2}(a_1,a_2) = \int_0^\infty e^{-px}x^\sigma E_{\nu_1}(a_1x)E_{\nu_2}(a_2x)\,dx$

When $p \neq 0$, unless new functions are introduced, it seems only to be possible to evaluate $I^2_{p,\sigma,\nu_1\nu_2}$ in special cases. Even when

$$\sigma, \nu_1, \nu_2 = s, n_1, n_2,$$

evaluation requires the introduction of the function $\psi(z)$, which is defined in the plane cut along the positive real axis from 1 to ∞ by

$$\psi(z) = -\int_0^z \frac{\log(1-\zeta)}{\zeta}\,d\zeta.$$

† The formula (5.3) is given by Kourganoff [see K 2 (2.8)], but he did not find the general formula (5.2).

‡ Cf. Glaisher, *Quart. J. of Math.* 18 (1882), 375 (9).

Kourganoff has shown that,† if $\mathbf{R}(a_1) > 0, \mathbf{R}(a_2) > 0, \mathbf{R}(p+a_1+a_2) > 0$, then

$$I^2_{p,0,1,1}(a_1, a_2) = V_p(a_1, a_2)$$

$$= -\tfrac{1}{6}\pi^2 + \log\frac{q}{a_1}\log\frac{q}{a_2} + \psi\left(\frac{q-a_1}{q}\right) + \psi\left(\frac{q-a_2}{q}\right), \tag{6.1}$$

where $q = p+a_1+a_2$.

When $\sigma, \nu_1, \nu_2 = s, n_1, n_2$, (3.6) gives

FORMULA III. *If* $|\arg a_1| < \pi, |\arg a_2| < \pi, \mathbf{R}(p+a_1+a_2) > 0, n_1 \geqslant 1, n_2 \geqslant 1, s \geqslant 1$, *then*

$$pI^2_{p,s,n_1n_2}(a_1, a_2) - (n_1+n_2+s-2)I^2_{p,s-1,n_1n_2}(a_1, a_2)$$
$$= -I^1_{p+a_1,s-1,n_2}(a_2) - I^1_{p+a_2,s-1,n_1}(a_1).$$

By the method (ii) of § 3 we also get

FORMULA IV. *If* $|\arg a_1| < \pi, |\arg a_2| < \pi, \mathbf{R}(p+a_1+a_2) > 0, n_1 \geqslant 2, n_2 \geqslant 1$, *then*

$$p(n_1-1)I^2_{p,0,n_1n_2}(a_1, a_2) + a_1(n_1+n_2-2)I^2_{p,0,n_1-1,n_2}(a_1, a_2)$$
$$= (p+a_1)I^1_{p+a_1,0,n_2}(a_2) + a_1 I^1_{p+a_2,0,n_1-1}(a_1).$$

FORMULA V. *If* $|\arg a_1| < \pi, |\arg a_2| < \pi, \mathbf{R}(p+a_1+a_2) > 0, n_1 \geqslant 2, n_2 \geqslant 2$, *then*

$$p^2(n_1-1)(n_2-1)I^2_{p,0,n_1n_2}(a_1, a_2) -$$
$$-a_1a_2(n_1+n_2-2)(n_1+n_2-3)I^2_{p,0,n_1-1,n_2-1}(a_1, a_2)$$
$$= p - a_1\{(n_1-1)p + (n_1+n_2-2)a_2\}I^1_{p+a_2,0,n_1-1}(a_1) -$$
$$-a_2\{(n_2-1)p + (n_1+n_2-2)a_1\}I^1_{p+a_1,0,n_2-1}(a_2).$$

By III and the results of § 4, I^2_{p,s,n_1n_2} can be reduced to $I^2_{p,0,n_1n_2}$ and by IV and V this can be reduced to $I^2_{p,0,1,1} = V_p(a_1, a_2)$. Thus, for example, from V and (4.2),

$$I^2_{p,0,2,2} = \frac{2a_1a_2}{p^2}V_p(a_1, a_2) + \frac{1}{p} - \frac{a_1}{p^2}(p+2a_2)I^1_{p+a_2,0,1}(a_1) -$$
$$-\frac{a_2}{p^2}(p+2a_1)I^1_{p+a_1,0,1}(a_2)$$
$$= \frac{2a_1a_2}{p^2}V_p(a_1, a_2) + \frac{1}{p} - \frac{a_1(p+2a_2)}{p^2(p+a_2)}\log\frac{q}{a_1} - \frac{a_2(p+2a_1)}{p^2(p+a_1)}\log\frac{q}{a_2},$$

where $q = p+a_1+a_2$, and this is K 2 (6.3).

The formulae III–V hold, of course, when s, n_1, n_2 are replaced by σ, ν_1, ν_2 provided that these are suitably restricted. III breaks down

† K 2 (4.16). Kourganoff gives his formula for real p, a_1, a_2, so I have extended it and I have corrected an error and regrouped the terms.

when $\sigma = 2-\nu_1-\nu_2$ and $\mathbf{R}(\nu_1) < 1$, $\mathbf{R}(\nu_2) < 1$, but, if we return to (3.5) [cf. the method used for (4.5)], with the aid of Theorem 1 we easily show that

$$pI^2_{p,2-\nu_1-\nu_2,\nu_1\nu_2}(a_1, a_2) = \Gamma(1-\nu_1)\Gamma(1-\nu_2)a_1^{\nu_1-1}a_2^{\nu_2-1} -$$

$$-\Gamma(2-\nu_1-\nu_2)(p+a_1+a_2)^{\nu_1+\nu_2-1}\left\{\frac{1}{a_2(1-\nu_1)}F\left(1, \nu_2; 2-\nu_1; -\frac{p+a_1}{a_2}\right) + \right.$$

$$\left. + \frac{1}{a_1(1-\nu_2)}F\left(1, \nu_1; 2-\nu_2; -\frac{p+a_2}{a_1}\right)\right\}, \qquad (6.2)$$

where $\mathbf{R}(a_1) > 0$, $\mathbf{R}(a_2) > 0$, $\mathbf{R}(p+a_1+a_2) > 0$.

BOUNDED INTEGRAL TRANSFORMS (II)

By I. J. GOOD (*London*)

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8. The present note is a continuation of a paper by I. J. Good and G. E. H. Reuter,† and the notation is the same as in that paper.

Every transformation T_ϕ (with domain L^2) is bounded and linear‡; every linear bounded transformation (with domain L^2) is a T_ϕ. The kernel ϕ of a linear bounded transformation is called a *standard kernel.* (This is a redefinition of a standard kernel and, if it is adopted, the previous Definition 3 becomes a theorem.) If T_1 and T_2 are bounded l.t.'s with bounds B_1 and B_2, then $T_2(T_1)$ is also a bounded l.t. with bound not greater than $B_1 B_2$. Let the kernels which give rise to T_1 and T_2 be ϕ_1 and ϕ_2. I shall denote the (standard) kernel which gives rise to $T_2(T_1)$ by $\phi_2.\phi_1$ and, if it requires a name, it may be called 'the operational product of ϕ_1 by ϕ_2'. (The use of this notation requires care. The operational square of ϕ is best denoted by $\phi.\phi$ and not by ϕ^2.) When we wish to bring the arguments x and t into evidence, we write $\phi_2.\phi_1(x,t)$ but not $\phi_2(x,t).\phi_1(x,t)$. This last expression would represent the ordinary product, but it is better to omit the full point for this sense.

The operational product of two standard kernels is determined by the following theorem.

THEOREM VI. *Let ϕ and ψ be standard kernels. Then the standard kernel $\psi.\phi$ is given by the equation*

$$\int_0^x \psi.\phi(y,t)\,dt = \int_0^\infty \psi(y,t)\phi^*(x,t)\,dt \qquad (11)$$

which holds for all non-negative x and y.

Before proving this theorem we notice that it includes Theorem V. For the kernel which gives rise to the identical transformation is $s_x(t)$, so that, if T_ψ is a left inverse of T_ϕ, then the left-hand side of (11) reduces to $\min(x,y)$.

† *Quart. J. of Math.* (Oxford), 19 (1948), 224–34.

‡ See, for example, E. Hille, *Functional Analysis and Semi-Groups* (New York, 1948), Theorem 2.13.9.

In order to prove Theorem VI I shall make use of the following

LEMMA. *If T is a bounded l.t. and ϕ is a standard kernel, then a necessary and sufficient condition for $T = T_\phi$ is that, whenever $a \geqslant 0$, we have $Ts_a(t) = \phi^*(a,t)$ for almost all t.*

Proof of the lemma. (i) *Necessity.* Let $a \geqslant 0$ be given and suppose that $T = T_\phi$. Then

$$\int_0^\infty s_a(t)\phi(x,t)\,dt = \int_0^a \phi(x,t)\,dt$$

$$= \int_0^x \phi^*(a,t)\,dt.$$

Therefore $Ts_a(t) = T_\phi s_a(t) = \phi^*(a,t)$ for almost all t.

(ii) *Sufficiency.* $Ts_a(t) = T_\phi s_a(t)$ for almost all t. Therefore, since T and T_ϕ are linear, $Ts(t) = T_\phi s(t)$ (almost everywhere) for any step-function $s(t)$. Therefore $Tf(t) = T_\phi s(t)$ (almost everywhere) for any given f which belongs to L^2, as required.

Proof of Theorem VI. By the definition of an adjoint kernel (which exists, by Theorem III), we have

$$\int_0^x \psi.\phi(y,t)\,dt = \int_0^y (\psi.\phi)^*(x,t)\,dt$$

$$= \int_0^y T_\psi T_\phi s_x(t)\,dt \qquad \text{(by the lemma)}$$

$$= \int_0^y T_\psi \phi^*(x,t)\,dt \qquad \text{(by the lemma)}$$

$$= \int_0^\infty \psi(y,t)\phi^*(x,t)\,dt$$

by the definition of T_ψ. (The theorem is also true for complex kernels, where

$$\int_0^x \phi(y,t)\,dt = \int_0^y \{\phi^*(x,t)\}\,dt,$$

the brackets $\{\}$ here denoting the complex conjugate.)

COROLLARIES

(i) $$\int_0^x \phi.\phi(y,t)\,dt = \int_0^\infty \phi(y,t)\phi^*(x,t)\,dt.$$

(ii) $\int_0^x \phi.\phi^*(y,t)\,dt = \int_0^\infty \phi(y,t)\phi(x,t)\,dt = \int_0^y \phi.\phi^*(x,t)\,dt.$

Thus $\phi.\phi^*$ is self-adjoint.

(iii) More generally the adjoint of $\psi.\phi$ is $\phi^*.\psi^*$.

(iv) If ϕ and ψ are bi-orthogonal, then $\psi.\phi^* = I$, where $I(x,t) = s_x(t)$.

9. The next theorem provides an alternative condition for a kernel to be standard.

THEOREM VII. *The function $\phi(x,t)$ $(0 \leqslant x < \infty;\ 0 \leqslant t < \infty)$ is a standard kernel if and only if there is a standard kernel Φ such that*

$$\int_0^\infty \phi(x,t)\phi(y,t)\,dt = \int_0^x \Phi(y,t)\,dt$$

for $0 \leqslant x < \infty$, $0 \leqslant y < \infty$. If this condition is satisfied, then

$$(\textit{modulus of } \phi)^2 = \textit{modulus of } \Phi\,; \textit{ and } T_\Phi \textit{ is self-adjoint.}$$

Proof. We know that, if ϕ is a standard kernel, then so is $\Phi = \phi.\phi^*$, and Φ is self-adjoint.

The first half of the theorem can be proved in half a page by showing that, if the conditions of Definition 3 are satisfied for Φ, then they are also satisfied for ϕ. (The proof involves one application of Schwarz's inequality.) I omit the details since Theorem VII is an immediate consequence of known facts about linear operators. If the moduli of ϕ and Φ are B and B', the proof (whose details have been omitted) shows only that $B^2 \leqslant B'$. But, by Theorem II, the modulus of ϕ^* is B, so that the modulus of $\Phi = \phi.\phi^*$ does not exceed B^2 (by the fifth sentence of § 8). So $B^2 = B'$.

10. Dr. H. Kober has informed me that in a thesis which he wrote some time ago he obtained the following condition for ϕ to give rise to a bounded l.t. with bound B. (I am largely indebted to Dr. G. E. H. Reuter for the method of presentation of both §§ 10, 11.) For any l_n, c_n $(n = 1, 2, \ldots, N;\ c_n > 0)$,

$$\int_0^\infty \Big[\sum_{n=1}^N l_n \phi(c_n,t)\Big]^2 dt \leqslant B^2 \sum_{m,n} l_m l_n \min(c_m, c_n). \tag{12}$$

The conditions (2) and (12) both state that

$$\int_0^\infty [T^*k(t)]^2\,dt \leqslant B^2 \int_0^\infty k^2(t)\,dt,$$

where $k(t)$ is an arbitrary step-function. For (2) take $k(t)$ as in § 9, while for (12) take

$$k(t) = \sum_{n=1}^{N} s_{c_n}(t).$$

The condition (12) has a simpler left-hand side, the condition (2) a simpler right-hand side. (Actually Dr. Kober expressed his results, quoted in §§ 10, 11, with slightly more generality, i.e. in the complex form.)

11. Dr. Kober has further pointed out the following facts.

(i) Closely allied to Theorem B is a theorem, say Theorem B′, expressible in the form

$$B_1 . B_2 . B_5 \to B_3 . B_4, \qquad B_4 . B_5 \to B_1 . B_2 . B_3.$$

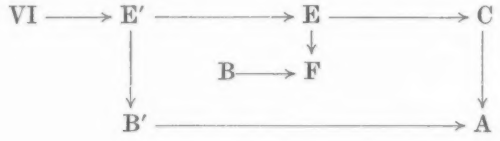
Theorem B′ (unlike Theorem B) includes Theorem A.

(ii) Suppose that ϕ and ψ are standard kernels. Then the transformations T_ϕ and T_ψ are inverse to each other if and only if

$$\int_0^\infty \phi^*(x,t)\psi(y,t)\,dt = \int_0^\infty \phi(x,t)\psi^*(y,t)\,dt = \min(x,y) \qquad (0 \leqslant x < \infty;\ 0 \leqslant y < \infty).$$

This theorem, which generalizes Theorem E, may be called Theorem E′. It is an immediate consequence of Theorem V.

(iii) Theorem E′ effectively includes B′. The implications are now



12. An iterative process for inverting bounded transformations

If T is a bounded linear transformation (with domain L^2) and if T^{-1} exists, then T^{-1} can be found by means of an iterative process. The process is similar to one which can be used for finding the reciprocal of a number or matrix, without the use of division.† I shall begin by finding U^{-1} where $U = T(T^*)$, after which T^{-1} can be determined as $T^*(U^{-1})$.

Let a be a real positive number. It will be necessary to restrict a, but the restriction is left unspecified for the moment. Let

$$U_0 = aI, \qquad U_{n+1} = 2U_n - U . U_n^2 \quad (n = 0, 1, 2, \ldots),$$

† See, for example, H. Hotelling, *Annals Math. Stat.* 14 (1943), 34.

where I now represents the identical transformation. We see at once by induction that U_n commutes with U. Let

$$V_n = U^{-1} - U_n.$$

Then
$$V_{n+1} = U.V_n^2,$$

and it follows by induction that

$$V_n = U^{-1}(I - aU)^{2^n}.$$

Therefore
$$|V_n| \leqslant |U^{-1}|.|I - aU|^{2^n},$$

where the modulus signs here denote the bounds of transformations. If we can select a so that $|I - aU| < 1$, it will follow that $|U^{-1} - U_n| \to 0$. In applications to linear integral equations this will enable us to obtain arbitrarily good approximations to the mean value in any given interval of the unknown function.

Now the bound of a transformation U for which $U^* = U$ is given by†

$$|U| = \overline{\text{bd}} \left| \int_0^\infty f.Uf\,dt \right|,$$

where the upper bound is taken over the set of functions f the squares of whose integrals are equal to 1. (This interpretation will be assumed for all upper and lower bounds to be mentioned in the present section.) But

$$\begin{aligned} \int_0^\infty f.Uf\,dt &= \int_0^\infty f.T^*(Tf)\,dt \\ &= \int_0^\infty (Tf)^2\,dt, \quad \text{by (5)}, \\ &\geqslant 0. \end{aligned}$$

It follows without difficulty that

$$|I - aU| = \max(|1 - ab_1|, |1 - ab_2|),$$

where

$$b_1 = \underline{\text{bd}} \int_0^\infty f.Uf\,dt \text{ (the lower bound of } U),$$

$$b_2 = \overline{\text{bd}} \int_0^\infty f.Uf\,dt = |U|.$$

Therefore by choosing a so that $0 < a < 2/b_1$ we have $|I - aU| < 1$, provided that $b_1 > 0$. Now this condition is satisfied if U has a bounded

† Stone, *Linear Transformations in Hilbert Space* (New York, 1932), 34.

inverse.† But, if T has a bounded inverse, then so also has T^*, by Theorems II and E′, and therefore so also has U.

In the above process perhaps the best value to take for a is $2/(b_1+b_2)$, since this minimizes $|I-aU|$. In fact it makes

$$|I-aU| = (b_2-b_1)/(b_2+b_1).$$

The convergence of the iterative process is therefore rapid if b_1/b_2 is not small, and is 'ultimately rapid' in any event.

In practice the calculation would probably be carried out without knowing b_1 and b_2, so that the 'best' value of a would not be known. It may then be necessary to try for a various integral powers of 2, such as 1, $\frac{1}{2}$, 2, 4, etc. The process could be generalized to complex kernels by means of trivial changes.

In the analogous process for inverting a non-singular matrix P, we first find Q^{-1}, where Q is the positive definite Hermitian matrix $\bar{P}'P$. ($\bar{P}'$ is the complex conjugate of the transpose of P.) The process is simpler than the one above. The numbers b_1 and b_2 become the smallest and largest *eigenvalues* of Q. If b_1 and b_2 are not known and a particular value of a is being tried, then the process will certainly not converge if at any stage the trace of Q_{n+2} is less than that of Q_{n+1}; where Q_0 is the scalar matrix aI and Q_n is defined by the recurrence relation $Q_{n+1} = 2Q_n - QQ_n^2$.

13. Summary

In § 8 the kernel is specified which gives rise to the product of two linear transformations of known kernels; §§ 9, 10 are concerned with forms of the condition for a kernel to be standard; § 11 is a condensed account of some remarks of Dr. Kober's; § 12 provides an iterative method for finding the inverse of a bounded transformation when the inverse exists.‡ The method may of course be regarded as one for solving linear integral equations. It is similar to a method for inverting a matrix. The convergence is rapid under specified conditions and is always ultimately rapid. The theories of Banach and Hilbert spaces are avoided as far as is convenient: there is, for example, no direct reference to ortho-normal sets.

I am indebted to Dr. F. Smithies for pointing out an error in a previous draft of § 12.

† See, for example, Hille, loc. cit., Theorem 2.13.1.

‡ Cf. A. T. Lonseth, *Proc. Berkeley Symp. Math. Statistics* (Berkeley and Los Angeles, 1949), 353–8.

A CLASS OF FOURIER KERNELS

By A. P. GUINAND (*Shrivenham*)

[Received 1 June 1949]

Introduction

If $K(x)$ satisfies appropriate conditions* and its Mellin transform

$$\mathfrak{K}(s) = \int_0^\infty K(x)x^{s-1}\,dx$$

satisfies $$\mathfrak{K}(s)\mathfrak{K}(1-s) = 1,$$

then $K(x)$ is a Fourier kernel.

If $$K(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos x,$$

then $$\mathfrak{K}(s) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Gamma(s)\cos\tfrac{1}{2}s\pi$$

and $$\mathfrak{K}(s)\mathfrak{K}(1-s) = \frac{2}{\pi}\Gamma(s)\Gamma(1-s)\cos\tfrac{1}{2}s\pi\sin\tfrac{1}{2}s\pi$$

$$= \frac{1}{\sin s\pi}\times 2\cos\tfrac{1}{2}s\pi\sin\tfrac{1}{2}s\pi = 1.$$

Thus the Fourier cosine kernel can be regarded as arising from the factorization $$\sin s\pi = 2\cos\tfrac{1}{2}s\pi\sin\tfrac{1}{2}s\pi.$$

A more general factorization is

$$\sin s\pi = 2^{2k-1}\prod_{r=0}^{2k-1}\sin(s+r)\frac{\pi}{2k}, \tag{1}$$

where k is a positive integer. Put

$$\mathfrak{K}(s) = 2^k(2\pi)^{-\frac{1}{2}}\Gamma(s)\prod_n\sin(s+n)\frac{\pi}{2k}, \tag{2}$$

where n runs through a set of k integers in $0 \leqslant n \leqslant 2k-1$ such that n and $2k-1-n$ are never both members of the set. Then

* E. C. Titchmarsh, *Fourier Integrals* (Oxford, 1937), Chapter VIII. The conditions cover all the examples considered here.

$$\mathfrak{K}(s)\mathfrak{K}(1-s)$$
$$= 2^{2k-1}\pi^{-1}\Gamma(s)\Gamma(1-s)\left\{\prod_n \sin(s+n)\frac{\pi}{2k}\right\}\left\{\prod_n \sin(1-s+n)\frac{\pi}{2k}\right\}$$
$$= \frac{2^{2k-1}}{\sin s\pi}\left\{\prod_n \sin(s+n)\frac{\pi}{2k}\right\}\left\{\prod_n \sin(s+2k-1-n)\frac{\pi}{2k}\right\} = 1,$$

since the two products together include all the factors in (1). Now, by repeated use of trigonometrical addition-formulae, (2) can be put in the form

$$\mathfrak{K}(s) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Gamma(s)\sum_{r=0}^{k} A_r \cos\left(\frac{rs\pi}{2k} + m_r\frac{\pi}{2k}\right), \tag{3}$$

where A_r are constants and m_r are integers. Further the functions

$$e^{-x\cos\theta}\cos(x\sin\theta+\phi), \qquad \Gamma(s)\cos(s\theta+\phi) \quad (0 \leqslant \theta \leqslant \tfrac{1}{2}\pi)$$

are a pair of Mellin transforms. Hence (3) corresponds to a Fourier kernel

$$K(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sum_{r=0}^{k} A_r\, e^{-x\cos(r\pi/2k)}\cos\left(x\sin\frac{r\pi}{2k} + m_r\frac{\pi}{2k}\right).$$

Examples. (i) If $k = 1$, we obtain the Fourier sine and cosine kernels

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\sin x, \qquad \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\cos x.$$

(ii) If $k = 2$, $n = 0$, 1, then

$$K(x) = \pi^{-\frac{1}{2}}(e^{-x} - \cos x + \sin x)$$

is a Fourier kernel. Similarly, if $k = 2$, $n = 2$, 3, then

$$K(x) = \pi^{-\frac{1}{2}}(e^{-x} + \cos x - \sin x)$$

is also a Fourier kernel. Further, if we put

$$\mathfrak{K}(s) = 4(2\pi)^{-\frac{1}{2}}\Gamma(s)\sin\tfrac{1}{4}(s+1)\pi\sin\tfrac{1}{4}(s+2)\pi,$$

and

$$\mathfrak{H}(s) = 4(2\pi)^{-\frac{1}{2}}\Gamma(s)\sin\tfrac{1}{4}s\pi\sin\tfrac{1}{4}(s+3)\pi,$$

then we get a pair of unsymmetrical kernels*

$$K(x) = \pi^{-\frac{1}{2}}(\cos x + \sin x + e^{-x})$$

and

$$H(x) = \pi^{-\frac{1}{2}}(\cos x + \sin x - e^{-x}).$$

* E. C. Titchmarsh, loc. cit., 214.

(iii) If $k = 3$, then the following Fourier kernels arise.*

$$(n = 0, 1, 2) \quad K(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \{2e^{-\frac{1}{2}x\sqrt{3}} \sin(\tfrac{1}{2}x + \tfrac{1}{6}\pi) - \cos x\};$$

$$(n = 0, 1, 3) \quad K(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \{e^{-\frac{1}{2}x\sqrt{3}} \cos(\tfrac{1}{2}x - \tfrac{1}{6}\pi) - \cos(x + \tfrac{1}{6}\pi)\};$$

$$(n = 0, 3, 4) \quad K(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \{e^{-\frac{1}{2}x\sqrt{3}} \sin(\tfrac{1}{2}x - \tfrac{1}{6}\pi) + \sin(x + \tfrac{1}{6}\pi)\};$$

$$(n = 1, 2, 5) \quad K(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \{-e^{-\frac{1}{2}x\sqrt{3}} \sin \tfrac{1}{2}x + \cos(x - \tfrac{1}{6}\pi)\};$$

$$(n = 2, 4, 5) \quad K(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \{e^{-\frac{1}{2}x\sqrt{3}} \cos \tfrac{1}{2}x - \sin(x - \tfrac{1}{6}\pi)\};$$

$$(n = 3, 4, 5) \quad K(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \{2e^{-\frac{1}{2}x\sqrt{3}} \cos(\tfrac{1}{2}x + \tfrac{1}{6}\pi) - \sin x\}.$$

* The choices $n = 0, 2, 4$ and $n = 1, 3, 5$ reduce to the ordinary Fourier sine and cosine kernels of (i).

ON THE BASIC BILATERAL HYPERGEOMETRIC SERIES ${}_2\Psi_2$

By W. N. BAILEY (*London*)

[Received 13 June 1949]

1. Introduction

THE formula†

$$1+\left\{(xy)\frac{(1-\alpha)}{(1-\beta x^2)}+\left(\frac{x}{y}\right)\frac{(1-\beta)}{(1-\alpha x^2)}\right\}+$$
$$+\left\{(xy)^2\frac{(1-\alpha)(x^2-\alpha)}{(1-\beta x^2)(1-\beta x^4)}+\left(\frac{x}{y}\right)^2\frac{(1-\beta)(x^2-\beta)}{(1-\alpha x^2)(1-\alpha x^4)}\right\}+\dots$$
$$=\frac{\Pi(xy,x^2)\Pi(x/y,x^2)\Pi(-x^2,x^2)\Pi(-\alpha\beta x^2,x^2)}{\Pi(\alpha xy,x^2)\Pi(\beta x/y,x^2)\Pi(-\alpha x^2,x^2)\Pi(-\beta x^2,x^2)}, \quad (1.1)$$

where $\Pi(a,b)=(1+a)(1+ab)(1+ab^2)\dots$, was given by Ramanujan and described by Hardy as 'a remarkable formula with many parameters'.

If we use the notation‡

$${}_r\Psi_r\left[\begin{matrix}\alpha_1,\alpha_2,\dots,\alpha_r;\\ \rho_1,\rho_2,\dots,\rho_r\end{matrix}z\right]=\sum_{n=-\infty}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\dots(\alpha_r)_n}{(\rho_1)_n(\rho_2)_n\dots(\rho_r)_n}z^n,$$

where

$$(a)_n=(1-a)(1-aq)\dots(1-aq^{n-1}),$$
$$(a)_{-n}=1/[(1-a/q)(1-a/q^2)\dots(1-a/q^n)]$$

and $|q|<1$, then (1.1) can be written in the form

$${}_1\Psi_1[1/\alpha;\beta x^2;-\alpha xy]$$
$$=\prod_{n=0}^{\infty}\left[\frac{(1-\alpha\beta x^{2n+2})(1+x^{2n+1}y)(1+x^{2n+1}/y)(1-x^{2n+2})}{(1-\alpha x^{2n+2})(1-\beta x^{2n+2})(1+\alpha x^{2n+1}y)(1+\beta x^{2n+1}/y)}\right], \quad (1.2)$$

where $q=x^2$, or, equivalently,

$${}_1\Psi_1[a;b;z]=\prod_{n=0}^{\infty}\left[\frac{(1-bq^n/a)(1-azq^n)(1-q^{n+1}/az)(1-q^{n+1})}{(1-q^{n+1}/a)(1-bq^n/az)(1-bq^n)(1-zq^n)}\right]. \quad (1.3)$$

When $b=q$, this becomes the well-known result

$${}_1\Phi_0[a;z]=\prod_{n=0}^{\infty}\left[\frac{1-azq^n}{1-zq^n}\right], \quad (1.4)$$

† See Hardy (**4**), § 12.12.2.

‡ Bailey (**2**).

and, if we replace q by q^2, then z by qz/a, let $a \to \infty$, and take $b = 0$, we get Jacobi's formula

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} z^n = \prod_{n=1}^{\infty} [(1-zq^{2n-1})(1-q^{2n-1}/z)(1-q^{2n})]. \quad (1.5)$$

It is easily seen that

$${}_1\Psi_1[a;b;z] = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} z^n + \sum_{n=1}^{\infty} \frac{(q/b)_n}{(q/a)_n} \left(\frac{b}{az}\right)^n,$$

and so we must assume, for convergence, that $|z| < 1$, $|b/az| < 1$.

In (2) I obtained the sum of a well-poised ${}_6\Psi_6$ which generalizes the sum of a well-poised ${}_6\Phi_5$. For bilateral series of the ordinary type Dougall† showed, as long ago as 1907, that the general series ${}_2H_2(1)$ could be summed in terms of gamma functions, but I obtained no analogous result for basic series. In this paper I give some transformations of the general series ${}_2\Psi_2$. Miss M. Jackson, to whom I communicated these results, has found that they have an interesting connexion with Lerch's transcendant.‡

2. Some transformations of the series ${}_2\Psi_2$

In the product $${}_1\Phi_0[a;z]\,{}_1\Psi_1[b;c;az]$$

the coefficient of z^n is

$$\sum_{r=0}^{\infty} \frac{(a)_r (b)_{n-r}}{(q)_r (c)_{n-r}} a^{n-r} = \frac{(b)_n a^n}{(c)_n} {}_2\Phi_1\left[\begin{matrix} a, q^{1-n}/c \\ q^{1-n}/b \end{matrix}; \frac{c}{ab}\right]$$

$$= \frac{(ab)_n}{(c)_n} \prod_{m=0}^{\infty} \left[\frac{(1-q^{m+1}/ab)(1-cq^m/b)}{(1-q^{m+1}/b)(1-cq^m/ab)}\right]$$

on using the basic analogue of Gauss's theorem. We therefore have the formula§

$${}_1\Phi_0[a;z]\,{}_1\Psi_1[b;c;az] = \prod_{m=0}^{\infty} \left[\frac{(1-q^{m+1}/ab)(1-cq^m/b)}{(1-q^{m+1}/b)(1-cq^m/ab)}\right] {}_1\Psi_1[ab;c;z], \quad (2.1)$$

where $|z| < 1$, $|az| < 1$, $|c/abz| < 1$.

From (2.1) it is evident that

$${}_1\Psi_1[b;c;az]\,{}_1\Psi_1[ab';c';z]$$

$$= \prod_{m=0}^{\infty} \left[\frac{(1-q^{m+1}/ab)(1-cq^m/b)(1-q^{m+1}/b')(1-c'q^m/ab')}{(1-q^{m+1}/b)(1-cq^m/ab)(1-q^{m+1}/ab')(1-c'q^m/b')}\right] \times$$

$$\times {}_1\Psi_1[b';c';az]\,{}_1\Psi_1[ab;c;z].$$

† Dougall (3). See also Bailey (2).

‡ Miss Jackson's paper is to be published in *J. of London Math. Soc.*

§ (2.1) can also be verified by using (1.3) and (1.4).

We now equate the terms independent of z on the two sides of this identity, and we obtain

$$
{}_2\Psi_2\left[\begin{matrix} b, q/c' \\ c, q/ab' \end{matrix};\ \frac{c'}{b'}\right]
$$

$$
= \prod_{m=0}^{\infty}\left[\frac{(1-q^{m+1}/ab)(1-cq^m/b)(1-q^{m+1}/b')(1-c'q^m/ab')}{(1-q^{m+1}/b)(1-cq^m/ab)(1-q^{m+1}/ab')(1-c'q^m/b')}\right]\times
$$

$$
\times {}_2\Psi_2\left[\begin{matrix} b', q/c \\ c', q/ab \end{matrix};\ \frac{c}{b}\right]. \quad (2.2)
$$

The terms in the series ${}_2\Psi_2$ can evidently be written in the reverse order, and so

$$
{}_2\Psi_2\left[\begin{matrix} a, b \\ c, d \end{matrix};\ z\right] = {}_2\Psi_2\left[\begin{matrix} q/c, q/d \\ q/a, q/b \end{matrix};\ \frac{cd}{abz}\right].
$$

When we use this relation (2.2) can be written in the form

$$
{}_2\Psi_2\left[\begin{matrix} \alpha, \beta \\ \gamma, \delta \end{matrix};\ z\right]
$$

$$
= \prod_{m=0}^{\infty}\left[\frac{(1-\delta q^{m+1}/\alpha\beta z)(1-\gamma q^m/\beta)(1-\alpha z q^m)(1-\delta q^m/\alpha)}{(1-q^{m+1}/\beta)(1-\gamma\delta q^m/\alpha\beta z)(1-\delta q^m)(1-zq^m)}\right]\times
$$

$$
\times {}_2\Psi_2\left[\begin{matrix} \alpha\beta z/\delta, \alpha \\ \alpha z, \quad \gamma \end{matrix};\ \frac{\delta}{\alpha}\right]. \quad (2.3)
$$

The formula (2.3) can be iterated, and we find that

$$
{}_2\Psi_2\left[\begin{matrix} \alpha, \beta \\ \gamma, \delta \end{matrix};\ z\right]
$$

$$
= \prod_{m=0}^{\infty}\left[\frac{(1-\alpha z q^m)(1-\beta z q^m)(1-\gamma q^{m+1}/\alpha\beta z)(1-\delta q^{m+1}/\alpha\beta z)}{(1-q^{m+1}/\alpha)(1-q^{m+1}/\beta)(1-\gamma q^m)(1-\delta q^m)}\right]\times
$$

$$
\times {}_2\Psi_2\left[\begin{matrix} \alpha\beta z/\gamma, \alpha\beta z/\delta \\ \alpha z, \quad \beta z \end{matrix};\ \frac{\gamma\delta}{\alpha\beta z}\right]. \quad (2.4)
$$

3. A transformation of a ${}_2\Psi_2$ into a series of well-poised type

I now obtain a different kind of transformation of the general ${}_2\Psi_2$ by using an artifice which I used in (**2**). We take a terminating ${}_8\Psi_8$ with $2n+1$ terms, multiply by a suitable constant so that the term farthest to the left becomes unity, and the series then becomes a well-poised ${}_8\Phi_7$ which can be transformed into a Saalschützian ${}_4\Phi_3$. This ${}_4\Phi_3$ can similarly be written as a ${}_4\Psi_4$ and, by making n tend to infinity, we obtain a transformation of the general ${}_2\Psi_2$.

The transformation† of a terminating well-poised ${}_8\Phi_7$ can be written as

$$\sum_{r=0}^{2n}\frac{(a)_r(q\sqrt{a})_r(-q\sqrt{a})_r(c)_r(d)_r(e)_r(f)_r(q^{-2n})_r}{(q)_r(\sqrt{a})_r(-\sqrt{a})_r(aq/c)_r(aq/d)_r(aq/e)_r(aq/f)_r(aq^{2n+1})_r}\frac{a^{2r}q^{2r+2nr}}{(cdef)^r}$$

$$=\prod_{m=1}^{\infty}\left[\frac{(1-aq^m)(1-aq^{m+2n}/e)(1-aq^{m+2n}/f)(1-aq^m/ef)}{(1-aq^{m+2n})(1-aq^m/e)(1-aq^m/f)(1-aq^{m+2n}/ef)}\right]\times$$

$$\times\sum_{r=0}^{2n}\frac{(aq/cd)_r(e)_r(f)_r(q^{-2n})_r}{(q)_r(ef/aq^{2n})_r(aq/c)_r(aq/d)_r}q^r.$$

Writing $r=n+s$ in both sides, and changing a, c, d, e, f into aq^{-2n}, cq^{-n}, dq^{-n}, eq^{-n}, fq^{-n}, we obtain the formula

$${}_8\Psi_8\left[\begin{matrix}aq^{-n}, q\sqrt{a}, -q\sqrt{a}, c, d, e, f, q^{-n}\\ q^n, \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, aq^{n+1}\end{matrix};\ \frac{a^2q^{2n+2}}{cdef}\right]$$

$$=\frac{(aq)_n(q/a)_n(aq/cd)_n(aq/ef)_n}{(q/c)_n(q/d)_n(aq/e)_n(aq/f)_n}\times$$

$$\times{}_4\Psi_4\left[\begin{matrix}aq^{n+1}/cd, e, f, q^{-n}\\ q^{n+1}, ef/aq^n, aq/c, aq/d\end{matrix};\ q\right]. \quad (3.1)$$

We now make n tend to infinity, the process being justified by a double application of Tannery's theorem to each side, and we obtain the transformation

$${}_2\Psi_2\left[\begin{matrix}e, & f\\ aq/c, & aq/d\end{matrix};\ \frac{aq}{ef}\right]$$

$$=\prod_{m=1}^{\infty}\left[\frac{(1-q^m/c)(1-q^m/d)(1-aq^m/e)(1-aq^m/f)}{(1-aq^m)(1-q^m/a)(1-aq^m/cd)(1-aq^m/ef)}\right]\times$$

$$\times\sum_{s=-\infty}^{\infty}\frac{(q\sqrt{a})_s(-q\sqrt{a})_s(c)_s(d)_s(e)_s(f)_s}{(\sqrt{a})_s(-\sqrt{a})_s(aq/c)_s(aq/d)_s(aq/e)_s(aq/f)_s}\left(\frac{a^3q}{cdef}\right)^s q^{s^2}. \quad (3.2)$$

As a limiting case of this transformation it may be noted that, when $d=a$ and c, e, f all tend to infinity, it becomes

$$\sum_{n=0}^{\infty}\frac{a^nq^{n^2}}{(q)_n}=\prod_{m=1}^{\infty}\left(\frac{1}{1-aq^m}\right)\sum_{s=0}^{\infty}\frac{(aq)_{s-1}}{(q)_s}(-1)^s(1-aq^{2s})a^{2s}q^{\frac{1}{2}s(5s-1)}, \quad (3.3)$$

which is the formula from which the Rogers–Ramanujan identities are usually derived.

It is evident that the series on the right of (3.2) is symmetrical in the

† Watson (**5**). See also Bailey (**1**), 69.

parameters c, d, e, and f, and so we can write down relations connecting series ${}_2\Psi_2$. The relations so obtained are equivalent to (2.3) and (2.4).

4. Conclusion

The origin of this paper was an attempt to find the sum of a series ${}_2\Psi_2$ analogous to the sum of a ${}_2H_2$ for series of the ordinary type. It will be noticed that, when $\gamma = q$ and $z = \delta/\alpha\beta$, (2.3) reduces to the basic analogue of Gauss's theorem, but I have been unable to obtain a corresponding sum for a ${}_2\Psi_2$. The formula analogous to (3.2) for series of the ordinary type involves a well-poised series ${}_5H_5(1)$ on the right which can be summed, but in (3.2) the factor q^{s^2} prevents the summation of the series on the right, and so there appears to be no sum for a ${}_2\Psi_2$ corresponding to that for a ${}_2H_2$.

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NON-HOMOGENEOUS BINARY QUADRATIC FORMS

By E. S. BARNES (*Cambridge*)

[Received 15 June 1949]

1. LET $f(x,y) = ax^2+bxy+cy^2$ be an indefinite binary quadratic form with real coefficients and discriminant $d = \Delta^2 = b^2-4ac > 0$. A well-known theorem of Minkowski states that, if x_0, y_0 are any real numbers, then there exist x, y satisfying $x \equiv x_0 \pmod 1$, $y \equiv y_0 \pmod 1$, or, say,

$$(x,y) \equiv (x_0,y_0) \pmod 1 \tag{1.1}$$

such that

$$|f(x,y)| \leqslant \tfrac{1}{4}\Delta; \tag{1.2}$$

the result being true with strict inequality if $f(x,y)$ does not represent zero for integer values of the variables, not both zero. I prove here a result of the form (1.2) but with Δ replaced by the function of the coefficients of $f(x,y)$:

$$\lambda = \lambda(f) = \max[|f(1,0)|, |f(0,1)|, \min\{|f(1,1)|, |f(1,-1)|\}]. \tag{1.3}$$

THEOREM 1. *For any real numbers x_0, y_0, there exist x, y satisfying* (1.1) *such that*

$$|f(x,y)| \leqslant \tfrac{1}{4}\lambda. \tag{1.4}$$

The result is true with strict inequality if (x_0, y_0) is not congruent (mod 1) *to one of the sets*

$$(0, \tfrac{1}{2}), (\tfrac{1}{2}, 0), (\tfrac{1}{2}, \tfrac{1}{2}). \tag{1.5}$$

COROLLARY. *The theorem remains true if λ is replaced by $\lambda(F)$ where $F(x,y)$ is any form equivalent to $\pm f(x,y)$ under integral unimodular transformation of the variables.*

The corollary is an immediate deduction, since (1.1) is transformed into a similar condition, and the sets (1.5) are merely permuted.

The relation between the estimates (1.2) and (1.4) is not an obvious one; in § 4, I show that Theorem 1 implies Davenport's improvement* of (1.2) for forms which do not represent zero, namely

$$|f(x,y)| \leqslant \tfrac{1}{4}\phi\left(\frac{f_1}{\Delta}\right)\Delta, \tag{1.6}$$

* H. Davenport, *Proc. K. Ned. Akad. Wet. Amsterdam*, 49 (1946), 815–21; 50 (1947), 378–89 and 484–91.

where

$$\phi(t) = \begin{cases} \sqrt{(1-4t^2)} & (0 < t \leqslant \frac{1}{4}\sqrt{2}), \\ \frac{1}{4}t^{-1} & (\frac{1}{4}\sqrt{2} \leqslant t \leqslant \frac{1}{2}), \\ t & (\frac{1}{2} \leqslant t < 1), \end{cases} \tag{1.7}$$

and f_1 is any value of $|f(x,y)|$ which corresponds to coprime integral values of x, y and which satisfies $0 < f_1 < \Delta$.

Among the most interesting indefinite binary quadratic forms are those of the Markoff series*

$$x^2+xy-y^2, \quad 2(x^2-2y^2), \quad 5x^2+11xy-5y^2, \ldots .$$

Davenport† has obtained precise values of the non-homogeneous minimum for the first three forms and a general estimate valid for all forms of the series, namely

$$|f(x,y)| \leqslant \tfrac{1}{4}\sqrt{5}\Delta. \tag{1.8}$$

In § 5, I use Theorem 1 to obtain the precise minimum of the first two forms and a stronger estimate than (1.8) for all other Markoff forms.

THEOREM 2. *Let Q be one of the Markoff numbers $1, 2, 5, 13, 29, \ldots$, and $f(x,y)$ a Markoff form with (homogeneous) minimum Q and discriminant $\Delta^2 = 9Q^2-4$. Then for any real numbers x_0, y_0 there exist x, y satisfying* (1.1) *for which*

$$|f(x,y)| \leqslant \tfrac{1}{4} \quad \textit{if} \quad Q = 1; \tag{1.9}$$

$$|f(x,y)| \leqslant 1 \quad \textit{if} \quad Q = 2; \tag{1.10}$$

$$|f(x,y)| \leqslant \tfrac{1}{4}\{2\sqrt{(5Q^2-4)}-3Q\} < \frac{2\sqrt{5}-3}{12}\Delta \quad \textit{if} \quad Q \geqslant 5. \tag{1.11}$$

Finally, in § 6, I give a direct application of Theorem 1 to the problem of the existence of a Euclidean algorithm in a real quadratic field.

I am indebted to Dr. J. H. H. Chalk, who suggested the original problem, and to Professor L. J. Mordell for valuable suggestions and discussions.

2. For the proof of Theorem 1 we need a simple lemma.

LEMMA. *If* $k \geqslant 1, \dfrac{1}{2k} \leqslant y \leqslant \frac{1}{2}$, *then*

$$(k^2y^2-\tfrac{1}{4})^{\frac{1}{2}}+\{k^2(y-1)^2-\tfrac{1}{4}\}^{\frac{1}{2}} \leqslant (k^2-1)^{\frac{1}{2}}, \tag{2.1}$$

$$(k^2y^2+\tfrac{1}{4})^{\frac{1}{2}}+\{k^2(y-1)^2+\tfrac{1}{4}\}^{\frac{1}{2}} \geqslant (k^2+1)^{\frac{1}{2}}. \tag{2.2}$$

The equality signs are necessary if and only if $y = \frac{1}{2}$.

* L. E. Dickson, *Studies in the Theory of Numbers* (Chicago, 1930), Chap. VII. Hereafter I quote this book as Dickson, *Studies*.

† See footnote on previous page.

Proof. We note that the conditions

$$k \geqslant 1, \quad \frac{1}{2k} \leqslant y \leqslant \tfrac{1}{2}$$

ensure that all the radicals are real. Now

$$\left.\begin{aligned} \frac{d}{dy}(k^2y^2 \pm \tfrac{1}{4})^{\frac{1}{2}} &= k^2y(k^2y^2 \pm \tfrac{1}{4})^{-\frac{1}{2}} \\ \frac{d}{dy}\{k^2(y-1)^2 \pm \tfrac{1}{4}\}^{\frac{1}{2}} &= k^2(y-1)\{k^2(y-1)^2 \pm \tfrac{1}{4}\}^{-\frac{1}{2}} \end{aligned}\right\}, \tag{2.3}$$

$$\left.\begin{aligned} \frac{d^2}{dy^2}(k^2y^2 \pm \tfrac{1}{4})^{\frac{1}{2}} &= \pm\tfrac{1}{4}k^2(k^2y^2 \pm \tfrac{1}{4})^{-\frac{3}{2}} \\ \frac{d^2}{dy^2}\{k^2(y-1)^2 \pm \tfrac{1}{4}\}^{\frac{1}{2}} &= \pm\tfrac{1}{4}k^2\{k^2(y-1)^2 \pm \tfrac{1}{4}\}^{-\frac{3}{2}} \end{aligned}\right\}. \tag{2.4}$$

By (2.3), the left-hand sides of (2.1), (2.2) have stationary values when $y+(y-1) = 0$, i.e. $y = \frac{1}{2}$, and by (2.4) this corresponds to a maximum of

$$(k^2y^2 - \tfrac{1}{4})^{\frac{1}{2}} + \{k^2(y-1)^2 - \tfrac{1}{4}\}^{\frac{1}{2}}$$

and to a minimum of

$$(k^2y^2 + \tfrac{1}{4})^{\frac{1}{2}} + \{k^2(y-1)^2 + \tfrac{1}{4}\}^{\frac{1}{2}}.$$

3. Proof of Theorem 1

(*a*) The result is obvious if $a = c = 0$; for then $\lambda = |b|$, and, if we choose x, y so that $|x| \leqslant \frac{1}{2}$, $|y| \leqslant \frac{1}{2}$, we have

$$|f(x,y)| = |bxy| \leqslant \tfrac{1}{4}|b|,$$

with inequality unless $(x,y) \equiv (\frac{1}{2}, \frac{1}{2}) \pmod 1$.

By interchanging x and y and writing $-f$ for f, if necessary, we may therefore assume that $a > 0$, $a \geqslant |c|$. Dividing through by a, we may take $a = 1$; and changing the sign of x, if necessary, we may suppose $b \geqslant 0$.

(*b*) We therefore confine ourselves hereafter to the 'canonical' form

$$f(x,y) = x^2 + bxy + cy^2, \quad \text{where} \quad b \geqslant 0, \ |c| \leqslant 1. \tag{3.1}$$

Then $$|f(1,-1)| = |1-b+c| \leqslant 1+b+c = |f(1,1)|,$$

and so $$\lambda(f) = \max\{1, |1-b+c|\}.$$

Since $b-c \leqslant 0$ implies $d \leqslant 0$, we have $b-c > 0$, and so

$$\lambda = \begin{cases} 1 & (b-c \leqslant 2), \\ -1+b-c & (b-c \geqslant 2). \end{cases} \tag{3.2}$$

We write, for convenience,

$$\lambda = t^2, \quad \tfrac{1}{4}b^2 - c = r^2, \quad k = r/t \quad \text{where} \quad r > 0,\ t > 0;$$

then

$$f(x,y) = (x+\tfrac{1}{2}by)^2 - r^2y^2.$$

Now choose y to satisfy $|y| \leqslant \frac{1}{2}$. We may then assume, by changing the signs of both x and y, if necessary, that $0 \leqslant y \leqslant \frac{1}{2}$.

(*c*) Suppose now that $r^2y^2 < \frac{1}{4}t^2$. Then the interval

$$-\tfrac{1}{2}by - (r^2y^2+\tfrac{1}{4}t^2)^{\frac{1}{2}} \leqslant x \leqslant -\tfrac{1}{2}by + (r^2y^2+\tfrac{1}{4}t^2)^{\frac{1}{2}} \tag{3.3}$$

has length $2(r^2y^2+\frac{1}{4}t^2)^{\frac{1}{2}} \geqslant t \geqslant 1$, and so we can always find in it a number $x \equiv x_0 \pmod 1$; while, if x belongs to the interval, we have

$$|x+\tfrac{1}{2}by| \leqslant (r^2y^2+\tfrac{1}{4}t^2)^{\frac{1}{2}},$$

i.e.

$$f(x,y) = (x+\tfrac{1}{2}by)^2 - r^2y^2 \leqslant \tfrac{1}{4}t^2.$$

Clearly

$$f(x,y) \geqslant -r^2y^2 > -\tfrac{1}{4}t^2,$$

and so

$$|f(x,y)| \leqslant \tfrac{1}{4}t^2.$$

Equality can occur here only if $f(x,y) = \frac{1}{4}t^2$, i.e. only if x must be chosen as one of the end points of the interval (3.3), i.e. only if the interval has length 1, which requires $y = 0$, $t = 1$, and x_0 congruent to one of the end points, which are now $\pm\frac{1}{2}$. Hence the equality sign is necessary only if $(x_0, y_0) \equiv (\frac{1}{2}, 0) \pmod 1$.

(*d*) Suppose next that $r^2y^2 \geqslant \frac{1}{4}t^2$, and so

$$\frac{1}{2k} \leqslant y \leqslant \tfrac{1}{2}, \quad k \geqslant 1.$$

Consider the four intervals

$$I_1 \quad [\mu_1 = -t\{k^2(y-1)^2+\tfrac{1}{4}\}^{\frac{1}{2}} - \tfrac{1}{2}b(y-1) \leqslant x \leqslant -t\{k^2(y-1)^2-\tfrac{1}{4}\}^{\frac{1}{2}} - \tfrac{1}{2}b(y-1) = \nu_1],$$

$$I_2 \quad [\mu_2 = t(k^2y^2-\tfrac{1}{4})^{\frac{1}{2}} - \tfrac{1}{2}by \leqslant x \leqslant t(k^2y^2+\tfrac{1}{4})^{\frac{1}{2}} - \tfrac{1}{2}by = \nu_2],$$

$$I_3 \quad [\mu_3 = -t(k^2y^2+\tfrac{1}{4})^{\frac{1}{2}} - \tfrac{1}{2}by + 1 \leqslant x \leqslant -t(k^2y^2-\tfrac{1}{4})^{\frac{1}{2}} - \tfrac{1}{2}by + 1 = \nu_3],$$

$$I_4 \quad [\mu_4 = t\{k^2(y-1)^2-\tfrac{1}{4}\}^{\frac{1}{2}} - \tfrac{1}{2}b(y-1) \leqslant x \leqslant t\{k^2(y-1)^2+\tfrac{1}{4}\}^{\frac{1}{2}} - \tfrac{1}{2}b(y-1) = \nu_4].$$

If x lies in I_2, we have

$$t(k^2y^2-\tfrac{1}{4})^{\frac{1}{2}} \leqslant x+\tfrac{1}{2}by \leqslant t(k^2y^2+\tfrac{1}{4})^{\frac{1}{2}},$$

$$-\tfrac{1}{4}t^2 \leqslant (x+\tfrac{1}{2}by)^2 - r^2y^2 \leqslant \tfrac{1}{4}t^2,$$

i.e.

$$|f(x,y)| \leqslant \tfrac{1}{4}t^2.$$

Similarly, if x lies in I_1, I_3, or I_4, we obtain

$$|f(x,y-1)| \leqslant \tfrac{1}{4}t^2, \quad |f(x-1,y)| \leqslant \tfrac{1}{4}t^2, \quad \text{or} \quad |f(x,y-1)| \leqslant \tfrac{1}{4}t^2$$

respectively; and clearly we have strictly $|f| < \frac{1}{4}t^2$ in all cases unless x is an end point of one of the intervals. Hence the conditions of the theorem will be fulfilled by at least one of the sets (x,y), $(x,y-1)$, $(x-1,y)$, provided that we can find a number $x \equiv x_0 \pmod 1$ lying in one of the four intervals. To show that this is always possible, I establish the following six inequalities:

(i) $\nu_2 \geqslant \mu_1$; with equality if and only if

$$y = \tfrac{1}{2},\ \nu_2 = \mu_1 = 0; \tag{3.4}$$

(ii) $\nu_1 \geqslant \mu_2+1$, if $b \geqslant 2$; with equality if and only if

$$y = \tfrac{1}{2},\ \nu_1 = \tfrac{1}{2},\ \mu_2 = -\tfrac{1}{2}; \tag{3.5}$$

(iii) $\nu_1 \geqslant \mu_2$; with equality if and only if

$$y = \tfrac{1}{2},\ \nu_1 = \mu_2 = 0; \tag{3.6}$$

(iv) $\nu_2 > \mu_3$; (3.7)

(v) $\nu_3 \geqslant \mu_4$, if $b \leqslant 2$, with equality if and only if

$$y = \tfrac{1}{2},\ \nu_3 = \mu_4 = \tfrac{1}{2}; \tag{3.8}$$

(vi) $\nu_4 > \mu_1+1$. (3.9)

(i) $\nu_2-\mu_1 = t[\{k^2(y-1)^2+\frac{1}{4}\}^{\frac{1}{2}}+(k^2y^2+\frac{1}{4})^{\frac{1}{2}}]-\frac{1}{2}b \geqslant t(k^2+1)^{\frac{1}{2}}-\frac{1}{2}b,$

by the lemma. Hence (3.4) is true if $t^2(k^2+1) \geqslant \frac{1}{4}b^2$, or $t^2 \geqslant \frac{1}{4}b^2-r^2 = c$, which follows from $|c| \leqslant 1 \leqslant t^2$. Equality occurs only if $t^2 = c$ and (2.2) is an equality; this requires $t^2 = c = 1$, $y = \frac{1}{2}$ and so

$$k^2 = r^2 = \tfrac{1}{4}b^2-1 \quad \text{and} \quad \mu_1 = \nu_2 = 0.$$

(ii) $\nu_1-\mu_2 = \frac{1}{2}b-t[(k^2y^2-\frac{1}{4})^{\frac{1}{2}}+\{k^2(y-1)^2-\frac{1}{4}\}^{\frac{1}{2}}] \geqslant \frac{1}{2}b-t(k^2-1)^{\frac{1}{2}},$

by the lemma. Hence, taking now $b \geqslant 2$, (3.5) is true provided that

$$(\tfrac{1}{2}b-1)^2 \geqslant t^2(k^2-1) = \tfrac{1}{4}b^2-c-t^2, \quad \text{i.e.} \quad t^2 \geqslant -1+b-c,$$

which follows from (3.2). Equality occurs only if $t^2 = -1+b-c$ and $y = \frac{1}{2}$ (giving equality in the lemma); then

$$\mu_2 = \tfrac{1}{2}(r^2-t^2)^{\frac{1}{2}}-\tfrac{1}{4}b = \tfrac{1}{2}(\tfrac{1}{2}b-1)-\tfrac{1}{4}b = -\tfrac{1}{2}.$$

Also, (3.6) is true provided that

$$\tfrac{1}{2}b \geqslant t(k^2-1)^{\frac{1}{2}}, \quad \text{i.e.} \quad \tfrac{1}{4}b^2 \geqslant r^2-t^2, \quad \text{i.e.} \quad t^2 \geqslant -c;$$

and in fact $t^2 \geqslant 1 \geqslant |c|$. Equality occurs only if $t^2 = -c = 1$, $y = \frac{1}{2}$, and so $k^2 = r^2 = \frac{1}{4}b^2+1$ and $\nu_1 = \mu_2 = 0$.

(iii) $$\begin{aligned}\nu_2-\mu_3 &= 2t(k^2y^2+\tfrac{1}{4})^{\frac{1}{2}}-1\\ &\geqslant \frac{2t}{\sqrt{2}}-1, \quad \text{since } y\geqslant\frac{1}{2k},\\ &> 0 \quad \text{since } t\geqslant 1,\end{aligned}$$

which proves (3.7).

(iv) $$\nu_3-\mu_4 = 1-\tfrac{1}{2}b-t[(k^2y^2-\tfrac{1}{4})^{\frac{1}{2}}+\{k^2(y-1)^2-\tfrac{1}{4}\}^{\frac{1}{2}}]$$
$$\geqslant 1-\tfrac{1}{2}b-t(k^2-1)^{\frac{1}{2}},$$

by the lemma. Hence, taking now $b\leqslant 2$, (3.8) is true if

$$(1-\tfrac{1}{2}b)^2\geqslant t^2(k^2-1),$$

which is proved as in (ii) above. Equality occurs only if

$$y=\tfrac{1}{2}, \qquad t^2=-1+b-c,$$

and then $$\nu_3 = -\tfrac{1}{2}(r^2-t^2)^{\frac{1}{2}}-\tfrac{1}{4}b+1=\tfrac{1}{2}.$$

(v) Finally, $$\begin{aligned}\nu_4-\mu_1-1 &= 2t\{k^2(y-1)^2+\tfrac{1}{4}\}^{\frac{1}{2}}-1\\ &\geqslant \frac{2t}{\sqrt{2}}-1, \quad \text{since} \quad |y-1|\geqslant\tfrac{1}{2},\\ &> 0,\end{aligned}$$

which proves (3.9).

The proof of the theorem is now easily completed. We distinguish the two cases $b\geqslant 2$, $b\leqslant 2$.

Suppose first that $b\geqslant 2$. We choose $x\equiv x_0 \pmod 1$ to satisfy $\mu_2<x\leqslant\mu_2+1$. Then either (*a*) $x<\nu_2$, so that $\mu_2<x<\nu_2$ and $|f(x,y)|<\frac{1}{4}t^2$; or (*b*) $x\geqslant\nu_2$, and so, by (3.4) and (3.5), $\mu_1\leqslant x\leqslant\nu_1$; hence $|f(x,y-1)|\leqslant\frac{1}{4}t^2$, with equality only if *either* $x=\mu_1=\nu_2$, whence by (3.4) $y=\frac{1}{2}$, $x=\nu_2=\mu_1=0$, *or* if $x=\nu_1=\mu_2+1$, whence by (3.5) $y=\frac{1}{2}$, $x=\nu_1=\mu_2+1=\frac{1}{2}$. Thus the choice of x ensures that either

$$|f(x,y)|<\tfrac{1}{4}\lambda \quad \text{or} \quad |f(x,y-1)|\leqslant\tfrac{1}{4}\lambda,$$

the equality sign being necessary only if $(x_0,y_0)\equiv(0,\frac{1}{2})$ or $(\frac{1}{2},\frac{1}{2})\pmod 1$.

Suppose next that $b\leqslant 2$. We choose $x\equiv x_0 \pmod 1$ to satisfy $\mu_1<x\leqslant\mu_1+1$. Then either (*a*) $\mu_1<x<\nu_1$, so that $|f(x,y-1)|<\frac{1}{4}t^2$; or (*b*) $x\geqslant\nu_1$. If now $x<\nu_2$, (3.6) shows that $\mu_2\leqslant x<\nu_2$, and so $|f(x,y)|\leqslant\frac{1}{4}t^2$, with equality only if $x=\mu_2=\nu_1$, whence by (3.6) $y=\frac{1}{2}$, $x=0$. If, however, $x\geqslant\nu_2$, then, by (3.7), $x>\mu_3$. Hence, if also $x<\nu_3$, we have $\mu_3<x<\nu_3$, and so $|f(x-1,y)|<\frac{1}{4}t^2$. If, however, $x\geqslant\nu_3$, (3.8) shows that $x\geqslant\mu_4$; also $x\leqslant\mu_1+1<\nu_4$, by (3.9), and so $\mu_4\leqslant x<\nu_4$. Hence $|f(x,y-1)|\leqslant\frac{1}{4}t^2$, with equality only if $x=\mu_4=\nu_3$, whence, by (3.8), $y=\frac{1}{2}$, $x=\frac{1}{2}$. Thus the choice of x ensures that (1.4) is satisfied

by at least one of the pairs (x, y), $(x, y-1)$, $(x-1, y)$, the equality sign being necessary only if $(x_0, y_0) \equiv (0, \frac{1}{2})$ or $(\frac{1}{2}, \frac{1}{2})$ (mod 1).

4. The result (1.6) may be written in the form

$$|f(x,y)| \leqslant \begin{cases} \frac{1}{4}\sqrt{(d-4f_1^2)} & \left(0 < f_1 \leqslant \dfrac{\Delta}{2\sqrt{2}}\right), \quad (4.1) \\ \frac{1}{4}(d/4f_1) & \left(\dfrac{\Delta}{2\sqrt{2}} \leqslant f_1 \leqslant \tfrac{1}{2}\Delta\right), \quad (4.2) \\ \frac{1}{4}f_1 & (\tfrac{1}{2}\Delta \leqslant f_1 < \Delta). \quad (4.3) \end{cases}$$

Since f_1 is a value of $|f|$ which corresponds to coprime integral values of x, y, we may suppose (after an integral unimodular transformation and writing $-f$ for f, if necessary) that

$$f(x,y) = ax^2+bxy-\gamma y^2, \quad (4.4)$$

where $a = f_1 > 0$. By a further transformation of the type $x = X+hY$, $y = \pm Y$, we may take $0 \leqslant b \leqslant a$, and then, since $b^2 \leqslant a^2 < d$, we have also $\gamma > 0$.

(i) Firstly, suppose $a \geqslant \frac{1}{2}\Delta$. Then

$$\gamma = \frac{d-b^2}{4a} \leqslant \frac{d}{4a} \leqslant a,$$

and $-a < -a+b+\gamma \leqslant \gamma$. Hence $|f(1,-1)| = |a-b-\gamma| \leqslant a$, and so $\lambda(f) = a$, and (4.3) follows from Theorem 1.

(ii) Next suppose that $\frac{1}{2}\Delta/\sqrt{2} \leqslant a \leqslant \frac{1}{2}\Delta$. Then

$$a \leqslant \frac{d}{4a}, \qquad \gamma = \frac{d-b^2}{4a} \leqslant \frac{d}{4a}.$$

Also $|a-\gamma| \leqslant d/4a$, and $0 \leqslant b \leqslant a \leqslant d/4a$. Hence

$$\min\{|f(1,\pm 1)|\} = |b-|a-\gamma|| \leqslant d/4a,$$

and so $\lambda(f) \leqslant d/4a$, which proves (4.2).

(iii) Finally suppose that $a \leqslant \frac{1}{2}\Delta/\sqrt{2}$. Choose $h \geqslant 0$ as the integer satisfying

$$ah^2+bh \leqslant \gamma < a(h+1)^2+b(h+1),$$

i.e.

$$0 \leqslant \gamma-ah^2-bh < 2ah+b+a. \quad (4.5)$$

Then, applying the integral unimodular transformation $x = X+hY$, $y = Y$ we obtain the equivalent form

$$g(x,y) = ax^2+(2ah+b)xy-(\gamma-ah^2-bh)y^2,$$

or, say,

$$g(x,y) = ax^2+\beta xy-\delta y^2,$$

where, by (4.5), $0 \leqslant \delta < a+\beta$, and clearly

$$\beta \geqslant b \geqslant 0. \quad (4.6)$$

We now show that $$\lambda(g) \leqslant \sqrt{(d-4a^2)} \tag{4.7}$$

which, with Theorem 1, establishes (4.1).

Since $a \leqslant \frac{1}{2}\Delta/\sqrt{2}$, $8a^2 \leqslant d$ and so

$$a \leqslant \tfrac{1}{2}\sqrt{(d-4a^2)}. \tag{4.8}$$

Also $$\delta < \sqrt{(d-4a^2)}. \tag{4.9}$$

For, if we had $\delta \geqslant \sqrt{(d-4a^2)}$, then $0 \leqslant \delta^2-(d-4a^2) = (2a-\delta)^2-\beta^2$. Also, by (4.8), $\delta \geqslant 2a$, and so $\delta-2a \geqslant \beta$, which contradicts (4.6).

If now $\delta > a$, then $d = \beta^2+4a\delta > \beta^2+4a^2$, and so $\beta < \sqrt{(d-4a^2)}$.

Hence $$0 < a+\beta-\delta < \beta < \sqrt{(d-4a^2)},$$

i.e. $$|a+\beta-\delta| = |g(1,1)| < \sqrt{(d-4a^2)}, \tag{4.10}$$

and (4.7) follows from (4.8), (4.9), and (4.10). But, if $\delta \leqslant a$, (4.8) gives $3a+\delta \leqslant 4a \leqslant 2\sqrt{(d-4a^2)}$. Hence

$$(3a+\delta)^2 \leqslant 4(d-4a^2),$$

$$25a^2-10a\delta+\delta^2 \leqslant 4\beta^2,$$

and so $$5a-\delta \leqslant 2\beta. \tag{4.11}$$

Accordingly

$$\begin{aligned}(d-4a^2)-(a-\beta-\delta)^2 &= 4a\delta-4a^2-(a-\delta)^2+2\beta(a-\delta)\\ &= (a-\delta)(2\beta-5a+\delta) \geqslant 0,\end{aligned}$$

by (4.11), since also $\delta \leqslant a$. Hence

$$|g(1,-1)| = |a-\beta-\delta| \leqslant \sqrt{(d-4a^2)}, \tag{4.12}$$

and (4.7) follows from (4.8), (4.9), and (4.12).

5. Proof of Theorem 2

If $Q = 1$, $f(x,y)$ is equivalent to x^2+xy-y^2, and $\lambda(f) = 1$. If $Q = 2$, $f(x,y)$ is equivalent to $2(x^2-2y^2)$, and $\lambda(f) = 4$. Thus (1.9) and (1.10) follow directly from (1.4).

Suppose then that $Q \geqslant 5$. Then* there corresponds to Q a set of integers of the form

$$2, 2, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_2, \alpha_2, \alpha_1, \alpha_1, 1, 1, \tag{5.1}$$

where each α_i is 1 or 2. The set (5.1) forms the period of partial quotients of the continued fraction for θ, where $x-\theta y$ is a factor of

$$f(x,y) = Qx^2-(P-Q')xy-P'y^2; \tag{5.2}$$

and $f(x,y)$ is equivalent to any Markoff form with minimum Q and

* Dickson, *Studies*, 98–9.

discriminant $9Q^2-4$. The integers P, Q, P', Q' are given in terms of the set (5.1) by the relations

$$\frac{P}{Q} = 2+\frac{1}{2+}\,\frac{1}{\alpha_1+}\,\ldots+\frac{1}{\alpha_1+}\,\frac{1}{1+}\,\frac{1}{1} = (2, 2, \alpha_1, \ldots, \alpha_1, 1, 1), \text{ say,} \quad (5.3)$$

$$\frac{P'}{Q'} = (2, 2, \alpha_1, \ldots, \alpha_1, 1), \quad (5.4)$$

whence it follows by elementary continued-fraction theory that*

$$P+Q' = 3Q, \quad (5.5)$$

$$PQ'-QP' = 1. \quad (5.6)$$

I first prove that

$$2P'-P \geqslant Q-Q'. \quad (5.7)$$

For the last four convergents to the continued fraction (5.3) are

$$\frac{2P'-P}{2Q'-Q}, \quad \frac{P-P'}{Q-Q'}, \quad \frac{P'}{Q'}, \quad \frac{P}{Q},$$

and so

$$L = \frac{2P'-P}{2Q'-Q} = (2, 2, \alpha_1, \alpha_1, \ldots, \alpha_2, \alpha_2, \alpha_1),$$

$$M = \frac{Q-Q'}{2Q'-Q} = (\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_1, \alpha_1, 2).\dagger$$

If every $\alpha_i = 2$, then $L = M$. Otherwise, suppose that α_r is the first α_i which equals 1. Then, since L and M are odd convergents,

$$\begin{aligned} M &< (\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_{r-1}, \alpha_{r-1}, \alpha_r, \alpha_r) \\ &= (\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_{r-1}, \alpha_{r-1}, 2) \\ &= (2, 2, \alpha_1, \alpha_1, \ldots, \alpha_{r-2}, \alpha_{r-2}, \alpha_{r-1}) \text{ (identically),} \\ &\leqslant L, \end{aligned}$$

since the odd convergents form an increasing sequence. Hence, in all cases, $L \geqslant M$, which proves (5.7).

Now, from (5.2),

$$|f(1,0)| = Q, \qquad |f(0,1)| = P', \qquad |f(1,-1)| = P-P'+Q-Q' \leqslant P',$$

by (5.7). Also $Q \leqslant P'$, since Q is the minimum of the form for integral x, y. Hence

$$\lambda(f) = P'. \quad (5.8)$$

It remains now to obtain an estimate for P' in terms of Q. To this end I prove

$$P'+Q \geqslant 4Q'. \quad (5.9)$$

* Dickson, *Studies*, pp. 98, 99.

† If $(a_1, a_2, \ldots, a_n) = \dfrac{p_n}{q_n}$, then $\dfrac{q_n}{q_{n-1}} = (a_n, a_{n-1}, \ldots, a_2)$.

For, by the same argument as above, we have

$$\frac{P'}{Q'} = (2, 2, \alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_1, 1), \qquad \frac{Q}{Q'} = (1, 1, \alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_1, 2).$$

Hence, if

$$\xi = (\alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_1, 1),$$

$$\eta = (\alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_1, 2) = (\alpha_1, \alpha_1, \ldots, \alpha_1, \alpha_1, 1, 1),$$

then

$$\frac{P'}{Q'} = (2, 2, \xi) = 2 + \frac{\xi}{2\xi+1}, \qquad \frac{Q}{Q'} = (1, 1, \eta) = 2 - \frac{1}{\eta+1},$$

and so

$$\frac{P'+Q}{Q'} = 4 + \frac{\xi}{2\xi+1} - \frac{1}{\eta+1} = 4 + \frac{\xi(\eta-1)-1}{(2\xi+1)(\eta+1)}. \tag{5.10}$$

Now, if every $\alpha_i = 1$, then $\eta = 1+1/\xi$. Hence $\xi(\eta-1)-1 = 0$ and (5.10) gives $(P'+Q)/Q' = 4$. Otherwise, suppose that

$$\alpha_1 = \alpha_2 = \ldots = \alpha_{r-1} = 1, \qquad \alpha_r = 2.$$

Then ξ and η, being odd convergents, are each greater than the preceding odd convergent

$$\zeta = (\alpha_1, \alpha_1, \ldots, \alpha_{r-1}, \alpha_{r-1}, \alpha_r) = (1, 1, \ldots, 1, 1, 2).$$

But also $\zeta = (1, 1, \ldots, 1, 1, 1, 1)$, and so is an even convergent to the continued fraction for $\frac{1}{2}(\sqrt{5}+1)$. Hence $\zeta > \frac{1}{2}(\sqrt{5}+1)$ and so

$$\xi > \tfrac{1}{2}(\sqrt{5}+1), \qquad \eta > \tfrac{1}{2}(\sqrt{5}+1), \qquad \xi(\eta-1) > 1,$$

and, by (5.10), $(P'+Q)/Q' > 4$. Thus, in all cases, $(P'+Q)/Q' \geqslant 4$, which proves (5.9).

Using (5.5), (5.6), we obtain

$$P'Q = PQ'-1 = 3QQ'-Q'^2-1. \tag{5.11}$$

Now clearly $Q' < Q$, and by (5.4), $P'/Q' < 3$. Hence $P' < 3Q$, $\frac{1}{4}(P'+Q) < Q$ and so, by (5.9),

$$Q' \leqslant \tfrac{1}{4}(P'+Q) < Q. \tag{5.12}$$

Since the expression $3Qz-z^2-1$ is an increasing function of z for $z < \frac{3}{2}Q$, (5.11) and (5.12) give

$$P'Q \leqslant 3Q\left(\frac{P'+Q}{4}\right) - \left(\frac{P'+Q}{4}\right)^2 - 1,$$

i.e.

$$(P'+3Q)^2 \leqslant 4(5Q^2-4),$$

and so

$$P' \leqslant 2\sqrt{(5Q^2-4)}-3Q,$$

whence, by (5.8),

$$\lambda(f) \leqslant 2\sqrt{(5Q^2-4)}-3Q. \tag{5.13}$$

The required result (1.11) now follows immediately from (5.13) and Theorem 1.

6. THEOREM 3. *The field $R(\sqrt{m})$ is Euclidean if* $m = 2, 3, 5, 6, 7, 13, 17, 21, 33, 37, 41, 57,$ *or* 73.*

An algebraic number-field is said to be *Euclidean* if, for any number κ of the field, there exists an integer ρ of the field such that

$$|\text{norm}(\rho-\kappa)| < 1.$$

The numbers of a real quadratic field $R(\sqrt{m})$, with basis $[1, \omega]$, are of the form $x+\omega y$ with rational x, y, the integers of the field corresponding to rational integral x, y. Then $\text{norm}(x+\omega y)$ is an indefinite quadratic form, say $f_m(x, y)$, where

$$f_m(x, y) = x^2-my^2 \quad \text{if} \quad m \equiv 2, 3 \pmod 4, \quad \omega = \sqrt{m};$$

$$f_m(x, y) = x^2+xy-\tfrac{1}{4}(m-1)y^2 \quad \text{if} \quad m \equiv 1 \pmod 4, \quad \omega = \tfrac{1}{2}(1+\sqrt{m}).$$

Thus $R(\sqrt{m})$ has a Euclidean algorithm if, given any (x_0, y_0), we can choose $(x, y) \equiv (x_0, y_0) \pmod 1$ such that $|f_m(x, y)| < 1$. Therefore, by Theorem 1, $R(\sqrt{m})$ is Euclidean if there exists a form $F_m(x, y)$ equivalent to $\pm f_m(x, y)$ for which *either* $\lambda(F_m) < 4$, *or* $\lambda(F_m) = 4$ and $|f_m(x, y)| < 1$ for some (x, y) congruent (mod 1) to $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$, or $(\frac{1}{2}, \frac{1}{2})$.

The following table exhibits a suitable $F_m(x, y)$ for each of the values of m stated in the theorem.

m	$f_m(x, y)$	$F_m(x, y)$	$\lambda(F_m)$
2	(1, 0, −2)	(1, 0, −2)	2
3	(1, 0, −3)	(1, 2, −2)	2
5	(1, 1, −1)	(1, 1, −1)	1
6	(1, 0, −6)	(1, 4, −2)	3
7	(1, 0, −7)	(1, 4, −3)	3
13	(1, 1, −3)	(1, 1, −3)	3
17	(1, 1, −4)	(1, 3, −2)	2
21	(1, 1, −5)	(1, 3, −3)	3
33	(1, 1, −8)	(2, 3, −3)	3
37	(1, 1, −9)	(3, 1, −3)	3
41	(1, 1, −10)	(1, 5, −4)	4
57	(1, 1, −14)	(2, 5, −4)	4
73	(1, 1, −18)	(3, 5, −4)	4

* The theorem is also true for $m = 11, 19, 29,$ and 97; but for no further values of $m > 0$, as has been proved recently by Davenport and Chatland. See *Bull. American Math. Soc.* 54 (1948), 662 and 828. For a detailed discussion of previous work, see L. Rédei, *Math. Ann.* 118 (1942), 588–608.

In order to establish strict inequality in the cases $m = 41, 57, 73$, for which $\lambda(F_m) = 4$, we observe that

$$f_{41}(2, -\tfrac{1}{2}) = \tfrac{1}{2}, \qquad f_{41}(\tfrac{1}{2}, 0) = \tfrac{1}{4}, \qquad f_{41}(\tfrac{3}{2}, \tfrac{1}{2}) = \tfrac{1}{2};$$
$$f_{57}(2, -\tfrac{1}{2}) = -\tfrac{1}{2}, \qquad f_{57}(\tfrac{1}{2}, 0) = \tfrac{1}{4}, \qquad f_{57}(\tfrac{3}{2}, \tfrac{1}{2}) = -\tfrac{1}{2};$$
$$f_{73}(2, \tfrac{1}{2}) = \tfrac{1}{2}, \qquad f_{73}(\tfrac{1}{2}, 0) = \tfrac{1}{4}, \qquad f_{73}(\tfrac{5}{2}, -\tfrac{1}{2}) = \tfrac{1}{2};$$

the three sets of values of (x, y) in each case being respectively congruent (mod 1) to $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, \frac{1}{2})$.

ANALOGUES FOR INTEGRAL FUNCTIONS OF CERTAIN THEOREMS ON POWER SERIES

By R. WILSON (*Swansea*)

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1. Introduction

In a classical paper† Pólya has derived certain properties of integral functions which are related to the characteristics of a power series in negative powers of the variable when this series is the Laplace transform of the integral function. In the simplest form the integral function is restricted to be of order 1, mean type, and in this special case the above-mentioned properties arise from the fact that the indicator diagrams of the integral function and the power series are images of one another in the real axis. Directions of strongest growth of the integral function are, to within reflection, the same as the singular directions of the power series; that is, radial directions which lead to a singularity on the circle of convergence. If we consider that series in positive powers which has the same coefficients as the given series, then the directions referred to are identical as are the indicator diagrams.

There are two ways in which the relevant properties of power series can be translated into properties of integral functions. One is to obtain the appropriate property for an integral function of order 1, mean type, as mentioned above, and then to construct proofs for the cases of any integral function of finite or infinite order. This, in fact, in a remarkable *tour de force*, is what Pólya has done. The other method is to generalize the Laplace transformation in such a way that the theory will apply to any integral function.

So far no such explicit generalization is known except for the case of an integral function of finite positive order and mean type.‡ For such a class of integral functions I give below three new theorems derived from theorems on power series due to Fabry and Braitzeff.

The general Laplace transformation here used‡ relates the series

$$f(z) = \sum_0^\infty a_n z^{-n-1} \tag{1}$$

to the integral function

$$F(z) = \sum_0^\infty \frac{a_n z^n}{\Gamma(\sigma+n\sigma)} = \sum_0^\infty c_n z^n \tag{2}$$

† (6), 610–40.

‡ (3), 7–11.

of order $\rho = \sigma^{-1}$ and type $h = r^{-\rho}$, where r is the radius of convergence of the power series

$$z^{-1}f(z^{-1}) = \sum_0^\infty a_n z^n. \tag{3}$$

As in the case of the ordinary Laplace-transform theory† the indicator diagram of (2) and the conjugate indicator diagram of (1) have the same vertices,‡ which are also the vertices of the indicator diagram of (3). Hence the radial directions leading to the singular points on the circle $|z| = r$ of (3) are identical with the directions of strongest growth of (2). In the case of an integral function of finite order these are directions such that in every angle, however small, containing a strong direction the order and type of the function is attained.

2. Theorems on power series

The initial theorems relating to the series (3) are set out below. Braitzeff's theorem is the most general and includes Fabry's general theorem, from which Fabry's special theorem can be deduced. For convenience these are set out in the reverse order to the above.

THEOREM 1.§ *If* $\lim_{n\to\infty} a_n/a_{n+1} = re^{i\alpha}$ *exists, then* $z = re^{i\alpha}$ *is a singular point for* $\sum_0^\infty a_n z^n$.

THEOREM 2.‖ *Let* σ_n *denote the number of changes of sign of*

$$\mathbf{R}(a_{n+\nu}\, e^{i(\nu\alpha-\gamma_n)})$$

in $-\theta n < \nu < \theta n$ $(0 < \theta < 1)$. *If there exists an increasing sequence of integers* n' *and a real sequence* $\gamma_{n'}$ *such that*

$$(a)\quad \lim_{n=n'\to\infty} [\mathbf{R}(a_n e^{-i\gamma_n})]^{1/n} = 1/r, \qquad (b)\quad \lim_{n=n'\to\infty} \sigma_n/n = 0,$$

then $z = re^{i\alpha}$ *is a singular point for* $\sum_0^\infty a_n z^n$.

THEOREM 3.†† *If in Theorem 2 condition* (*b*) *is replaced by the more general condition* $\liminf_{n=n'\to\infty} \sigma_n/(2\theta n) = \beta$, *then at least one point* $z = re^{i\phi}$ *is singular for* $\sum_0^\infty a_n z^n$, *where* $-\pi\beta \leqslant \phi - \alpha \leqslant \pi\beta$.

Further, if there is only one singular point on $|z| = r$, *then either* $\phi = \alpha + \pi\beta$ *or* $\phi = \alpha - \pi\beta$.

† (6), 571–610. ‡ (3), (5). § (1), 377–9.
‖ (1), (2), (4). †† (4), 26–8.

3. Theorems on integral functions

By use of the relations between (1), (2), and (3) we obtain the following analogue of Theorem 1.

THEOREM 1*. *If* $\sum_0^\infty c_n z^n$ *is an integral function of order* ρ *and if* $\lim_{n\to\infty} n^{1/\rho} c_{n+1}/c_n = ae^{i\alpha}$ *exists, then the type of* $\sum_0^\infty c_n z^n$ *is* a^ρ/ρ *and* $e^{-i\alpha}$ *is a direction of strongest growth of* $\sum_0^\infty c_n z^n$.

We consider the function

$$z^{-1}f(z^{-1}) = \sum_0^\infty a_n z^n = \sum_0^\infty \Gamma(\sigma+n\sigma)c_n z^n.$$

For this series

$$\lim_{n\to\infty} a_n/a_{n+1} = \lim_{n\to\infty} \frac{\Gamma(\sigma+n\sigma)}{\Gamma(2\sigma+n\sigma)} \frac{c_n}{c_{n+1}} = \lim_{n\to\infty} \left(\frac{\rho}{n}\right)^{1/\rho} \frac{n^{1/\rho}}{ae^{i\alpha}}$$
$$= \rho^{1/\rho} e^{-i\alpha}/a = re^{-i\alpha},$$

where r is the radius of convergence of $\sum_0^\infty a_n z^n$.

Hence, by Theorem 1, $z = re^{-i\alpha}$ is a singular point of $z^{-1}f(z^{-1})$ on its circle of convergence, and so $e^{-i\alpha}$ is a direction of strongest growth for $F(z)$. Further, the type of $F(z)$ is $r^{-\rho} = a^\rho/\rho$.

Corresponding to Theorem 2 we have the following

THEOREM 2*. *Suppose that* $\sum_0^\infty c_n z^n$ *is an integral function of order* ρ *and type* h, *each finite and positive; let* σ_n *denote the number of changes of sign of* $\mathbf{R}(c_{n+\nu} e^{i(\nu\alpha-\gamma_n)})$ *in* $-\theta n < \nu < \theta n$ $(0 < \theta < 1)$. *If there exists an increasing sequence of integers* n' *and a real sequence* $\gamma_{n'}$ *such that*

$$(a) \quad \lim_{n=n'\to\infty} n[\mathbf{R}(c_n e^{-i\gamma_n})]^{\rho/n} = he\rho, \qquad (b) \quad \lim_{n=n'\to\infty} \sigma_n/n = 0,$$

then $e^{i\alpha}$ *is a direction of strongest growth of* $\sum_0^\infty c_n z^n$.

Here $$z^{-1}f(z^{-1}) = \sum_0^\infty a_n z^n = \sum_0^\infty \Gamma(\sigma+n\sigma)c_n z^n$$

is a series with radius of convergence $h^{-1/\rho}$. Hence, for this series, condition (a) becomes

$$\lim_{n=n'\to\infty} [\mathbf{R}(a_n e^{-i\gamma_n})]^{1/n} = h^{1/\rho},$$

while condition (b) remains unaltered when applied to $\mathbf{R}(a_{n+\nu} e^{i(\nu\alpha-\gamma_n)})$.

Thus, from Theorem 2, $z = h^{-1/\rho}e^{i\alpha}$ is a singular point of $z^{-1}f(z^{-1})$ and therefore $e^{i\alpha}$ is a direction of strongest growth for $F(z)$.

A similar proof making use of Theorem 3 leads to the following

THEOREM 3*. *If in Theorem 2* condition (b) is replaced by the more general condition* $\liminf_{n=n'\to\infty} \sigma_n/(2\theta n) = \beta$, *then at least one direction* $e^{i\phi}$ *is a direction of strongest growth of* $\sum_0^\infty c_n z^n$, *where* $-\pi\beta \leqslant \phi - \alpha \leqslant \pi\beta$.

Further, if there is only one strongest direction, then either $\phi = \alpha + \pi\beta$ *or* $\phi = \alpha - \pi\beta$.

It is of interest to note that the two forms of condition (a) in Theorems 2 and 2* are satisfied by $z^{-1}f(z^{-1})$ and $F(z)$ for the same sequence (n').

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ON THE CRITICAL DETERMINANTS OF CYLINDERS

By H. DAVENPORT and C. A. ROGERS (*London*)

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LET K be a star domain in the xy-plane, and let $C = C(K)$ be the three-dimensional cylinder consisting of all points (x, y, z) for which (x, y) is in K and $|z| \leqslant 1$. Let $\Delta(K)$ and $\Delta(C)$ denote the critical determinants of K and C. It is easily seen that $\Delta(C) \leqslant \Delta(K)$, and it is known that this is true with equality when K is convex. Rogers* has recently given an example of a star domain K for which $\Delta(C) < \Delta(K)$. The object of the present note is to give an example for which the ratio $\Delta(C)/\Delta(K)$ is arbitrarily small.

Let θ be an irrational number, and m a positive integer. We consider the system of points in the xy-plane, given by

$$x = u+\theta w, \qquad y = v,$$

where u, v, w take all integral values except 0, 0, 0, subject to $|w| < m$. For each such point, say P, we cut out of the plane a thin angle, with vertex at P, containing the continuation of the line OP. We shall take K to be the remainder of the xy-plane after all these thin angles have been removed.

For a more precise specification, let δ be a small positive number, which will be chosen later so as to depend only on θ and m. We can define the thin angle mentioned above by

$$x = \lambda(u+\theta w)+\mu v, \qquad y = \lambda v-\mu(u+\theta w), \tag{1}$$

where λ, μ take all real values satisfying

$$\lambda > 1, \qquad |\mu| < \delta(\lambda-1). \tag{2}$$

We define K to consist of all points (x, y) which are *not* representable in the form (1), where u, v, w are integers, other than 0, 0, 0, with $|w| < m$, and where λ and μ satisfy (2).

Plainly K is symmetrical about O. Also K is bounded. For, if we take $w = 0$, the equations (1) give

$$\lambda = \frac{xu+yv}{u^2+v^2}, \qquad \mu = \frac{xv-yu}{u^2+v^2}.$$

* *Quart. J. of Math.* (Oxford), 20 (1949), 45–7. References are given there.

If x^2+y^2 is sufficiently large, and we choose $u = [\frac{1}{2}x]$, $v = [\frac{1}{2}y]$, these values of λ and μ satisfy (2), and so such a point is outside K. Finally, it is clear from the construction that any radius vector from O meets the boundary of K in exactly one point, this point varying continuously with the radius vector. Hence K is a bounded star domain.

The special three-dimensional lattice defined by

$$x = u+\theta w, \qquad y = v, \qquad z = w/m, \tag{3}$$

where u, v, w are integral variables, has no point other than O in the interior of the cylinder $C(K)$. For such a point would have $|w| < m$, and so x, y would be of the form (1) with $\lambda = 1$, $\mu = 0$. But then the point (x, y) would be a boundary point, and not an interior point, of K. Since the determinant of the lattice (3) is $1/m$, we have

$$\Delta(C) \leqslant 1/m. \tag{4}$$

We proceed to prove now that

$$\Delta(K) \geqslant 1, \tag{5}$$

provided that δ is sufficiently small.* By (4) and (5) we shall then have the result we require, since m may be taken as large as we please.

To prove (5) we have to show that every lattice in the xy-plane, of determinant less than 1, has a point other than O in the interior of K. We therefore assume that we have a lattice of determinant less than 1, with the property that every lattice point other than O is of the form (1), subject to the conditions on u, v, w, and subject to (2), with the modification that equalities are now admitted there.

We first obtain some inequalities for the numbers u, v, w, λ, μ corresponding to a lattice point (x, y) other than O. The equations (1) are satisfied, and

$$\lambda \geqslant 1, \qquad |\mu| \leqslant \delta(\lambda-1). \tag{6}$$

Hence

$$|v| = \frac{|\lambda y+\mu x|}{\lambda^2+\mu^2} \leqslant \frac{\lambda|y|+\delta(\lambda-1)|x|}{\lambda^2}$$

$$= \begin{cases} |y|-(\lambda|y|-\delta|x|)(\lambda-1)\lambda^{-2}, \\ \delta|x|\lambda^{-1}+(\lambda|y|-\delta|x|)\lambda^{-2}; \end{cases}$$

whence

$$|v| \leqslant \max(|y|, \delta|x|). \tag{7}$$

Similarly

$$|u+\theta w| \leqslant \max(|x|, \delta|y|). \tag{8}$$

* It follows from (5) that in fact $\Delta(K) = 1$, since any point with integral coordinates, other than O, is of the form (1) with $w = 0$, $\lambda = 1$, $\mu = 0$, and so is not inside K.

Now we can choose a positive number ϵ, depending only on θ and m (and independent of δ), so small that $|u+\theta w| > \epsilon$ for all integers u, w (not both zero) with $|w| < m$. As (x, y) is not O, it is clear that

$$\max(|u+\theta w|, |v|) > \epsilon.$$

Thus
$$1 \leqslant \lambda = \frac{(u+\theta w)x+vy}{(u+\theta w)^2+v^2} \leqslant \frac{|x|+|y|}{\epsilon},$$

and consequently
$$|\mu| < \delta\epsilon^{-1}(|x|+|y|). \tag{9}$$

Since the determinant of the lattice is less than 1, there is a lattice point other than O satisfying $|x| < 1$, $|y| < 1$. If this point is represented in the form (1), we have, by (7),

$$|v| < 1.$$

Thus $v = 0$, and

$$|y| = |\mu(u+\theta w)| \leqslant \delta\lambda|u+\theta w| = \delta|x|.$$

Further, as u and w may not both vanish, it follows from our choice of ϵ that

$$|x| = \lambda|u+\theta w| > \epsilon.$$

Hence there is a lattice point (ξ_1, η_1) with

$$\epsilon < \xi_1 < 1, \qquad |\eta_1| < \delta. \tag{10}$$

Again there is a lattice point other than O satisfying $|x| < \epsilon, |y| < \epsilon^{-1}$. It follows by (8) that for this point

$$|u+\theta w| < \epsilon,$$

provided that δ is sufficiently small. Thus, by our choice of ϵ, it follows that $u = w = 0$ and

$$|x| = |\mu v| < \delta\lambda|v| = \delta|y| < \delta\epsilon^{-1}.$$

Further, as v cannot vanish,

$$|y| = \lambda|v| \geqslant 1.$$

Hence there is a lattice point (ξ_2, η_2) with

$$0 \leqslant \xi_2 < \delta\epsilon^{-1}, \qquad 1 \leqslant |\eta_2| < \epsilon^{-1}. \tag{11}$$

Since the lattice point $(\xi_1-\xi_2, \eta_1-\eta_2)$ is of the form (1), we have

$$\xi_1-\xi_2 = \lambda_0(u_0+\theta w_0)+\mu_0 v_0, \qquad \eta_1-\eta_2 = \lambda_0 v_0-\mu_0(u_0+\theta w_0), \tag{12}$$

where u_0, v_0, w_0 are integers with $|w_0| < m$, and

$$\lambda_0 \geqslant 1, \qquad |\mu_0| \leqslant \delta(\lambda_0-1). \tag{13}$$

Applying (7), (8), (9) to the point $(\xi_1-\xi_2, \eta_1-\eta_2)$, and noting that these coordinates are bounded by (10) and (11), we have

$$u_0+\theta w_0 = O(1), \qquad v_0 = O(1), \qquad \mu_0 = O(\delta), \tag{14}$$

using the O-notation with the convention that the implied constants depend only on θ and m. By (14), (10), (11), (12),

$$\xi_1 = \lambda_0(u_0+\theta w_0)+O(\delta), \qquad \eta_2 = -\lambda_0 v_0+O(\delta). \tag{15}$$

Note that, as $|\eta_2| \geqslant 1$, it follows that $|v_0| \geqslant 1$. Also, if w_0 were zero, we should have

$$|u_0| \leqslant \max(|\xi_1-\xi_2|, \delta|\eta_1-\eta_2|) = |\xi_1-\xi_2| < 1$$

from (8), (10), (11); so that u_0 would be zero. But then $\xi_1 = O(\delta)$ by the first half of (15), and this would contradict the first half of (10), supposing δ sufficiently small. Thus $w_0 \neq 0$.

Now consider the lattice point $(mv_0\xi_1-\xi_2, mv_0\eta_1-\eta_2)$. We have

$$mv_0\xi_1-\xi_2 = \lambda(u+\theta w)+\mu v, \qquad mv_0\eta_1-\eta_2 = \lambda v-\mu(u+\theta w), \tag{16}$$

where u, v, w are integers with $|w| < m$, and

$$\lambda \geqslant 1, \qquad |\mu| \leqslant \delta(\lambda-1). \tag{17}$$

As before, $\quad u+\theta w = O(1), \qquad v = O(1), \qquad \mu = O(\delta),$

and

$$mv_0\xi_1 = \lambda(u+\theta w)+O(\delta), \qquad \eta_2 = -\lambda v+O(\delta). \tag{18}$$

Combining (15) and (18),

$$\xi_1\eta_2 = \lambda_0(u_0+\theta w_0)(-\lambda v)+O(\delta),$$

$$mv_0\xi_1\eta_2 = \lambda(u+\theta w)(-\lambda_0 v_0)+O(\delta).$$

Thus

$$\lambda\lambda_0 v_0(u+\theta w) = \lambda\lambda_0 mv_0 v(u_0+\theta w_0)+O(\delta).$$

As $\lambda \geqslant 1$, $\lambda_0 \geqslant 1$, $|v_0| \geqslant 1$, we may divide this by $\lambda\lambda_0 v_0$ to give

$$(mvu_0-u)+(mvw_0-w)\theta = O(\delta). \tag{19}$$

Now $|w| < m$ and $|mvw_0| \geqslant m$. Hence

$$0 < |mvw_0-w| = O(1). \tag{20}$$

Since θ is irrational and m, v, u_0, u, w_0, w are integers, the inequalities (19) and (20) cannot hold if δ is chosen to be less than a certain positive number depending only on θ and m. This contradiction establishes (5) for such values of δ.

ON GROUP EXTENSIONS WITH OPERATORS

By J. H. C. WHITEHEAD (*Oxford*)

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1. Introduction

In the study of groups with operators, which has important applications to topology, we are led to consider group extensions with operators in a separate group. More precisely, we consider two groups E, Y, each of which admits a group X as a group of operators, and an operator homomorphism, $\phi: E \to Y$, onto Y. Subject to these conditions we describe (E, ϕ) as an *X-extension*† of Y by G, where $G = \phi^{-1}(1)$. Since $\phi x = x\phi$, where $x \in X$ is any operator, it follows that $xG = G$. Therefore G also admits X as a group of operators, $x: G \approx G$ being the automorphism induced by $x: E \approx E$. If Y, G, and X operating on Y, G are given in the first place, an X-extension of Y by G will mean one in which X operates in the given way on Y and G, subject to the above conditions.

Let (E, ϕ) and (E', ϕ') be two X-extensions of Y by G. We describe them as *equivalent* if and only if there is an operator isomorphism, $\psi: E \approx E'$, such that $\phi'\psi = \phi$ and $\psi g = g$ for each $g \in G$. This is obviously an equivalence relation between (E, ϕ) and (E', ϕ').

We take G to be Abelian and are thus led to a 'vector' cohomology theory with partial co-boundary operators of the kind considered in (**4**). I show how this theory may be applied to the classification of X-extensions and also to the study of Y-kernels, as defined in (**2**), with operators in X.

I recall the basic definitions in the algebraic cohomology theory. Let A be an additive Abelian group, which admits a multiplicative group W as a group of operators, both from the left and the right, subject to the condition‡ $w(aw') = (wa)w'$. Then $C^n(W, A)$ is the additive co-chain group, which consists of all functions of n variables in W with values in A. The co-boundary operator,

$$\delta: C^n(W, A) \to C^{n+1}(W, A), \tag{1.1}$$

† Thus (E, ϕ) may be defined by discarding the condition that G is a direct summand of E in the definition of a Y-enlargement [see reference (**3**)]. For an account of the cohomology theory of group extensions see (**1**), (**2**), and (**3**). Notice that I follow Eilenberg and MacLane in referring to (E, ϕ) as an extension of Y by G, not of G by Y.

‡ Small italic letters in formulae will usually denote elements in the groups denoted by the corresponding capitals.

is defined by

$$(\delta c)(w_1,\ldots,w_{n+1}) = w_1 c(w_2,\ldots,w_{n+1}) + \\ + \sum_{i=1}^{n} (-1)^i c(w_1,\ldots,w_{i-1},w_i w_{i+1},w_{i+2},\ldots,w_n) + \\ + (-1)^{n+1} c(w_1,\ldots,w_n) w_{n+1}, \quad (1.2)$$

where $c \in C^n(W,A)$. If W operates only from the left, then δ is given by the same rule with simple right-hand operators (i.e. $aw = a$). It is proved in (**1**) that $\delta\delta = 0$ and the theory is mainly concerned with the cohomology groups,

$$H^n(W,A) = Z^n(W,A) - \delta C^{n-1}(W,A),$$

where $Z^n(W,A)$, the group of 'n-dimensional' co-cycles, is the kernel of (1.1) and $C^{-1}(W,A) = 0$.

2. Vector cohomology groups

Let G be an additive Abelian group, which admits an arbitrary multiplicative group Y as a group of operators. I shall use $y.g$ to denote the image of g under the operator y. Let G and Y both admit a multiplicative group X as a group of operators. I shall use xg, xy, without full points, to denote the images of g, y under the operator x. The symbol $xy.g$ will always denote $(xy).g$ and I insist that

$$x(y.g) = xy.xg. \quad (2.1)$$

Notice that

$$\left.\begin{aligned} x1.g &= 1.g = g \\ xy.(xy'.g) &= (xyxy').g = x(yy').g \end{aligned}\right\}, \quad (2.2)$$

where $1 \in Y$ is the unit element.

Let m, $n \geqslant 0$ and let X operate on the co-chain group $C^n(Y,G)$, from the left and from the right, according to the rules

$$\left.\begin{aligned} (xc)(y_1,\ldots,y_n) &= xc(y_1,\ldots,y_n) \\ (cx)(y_1,\ldots,y_n) &= c(xy_1,\ldots,xy_n) \end{aligned}\right\}, \quad (2.3)$$

where $c \in C^n(Y,G)$. Let Y operate from the left on $C^m(X,G)$ according to the rule

$$(yc)(x_1,\ldots,x_m) = x_1\ldots x_m\, y.c(x_1,\ldots,x_m), \quad (2.4)$$

where $c \in C^m(X,G)$. It follows from (2.2), with $x = x_1\ldots x_m$, and from (2.4) that $1c = c$ and that

$$\begin{aligned} \{y(y'c)\}(x_1,\ldots,x_m) &= xy.\{(y'c)(x_1,\ldots,x_m)\} \\ &= xy.\{xy'.c(x_1,\ldots,x_m)\} \\ &= x(yy').c(x_1,\ldots,x_m) \\ &= \{(yy')c\}(x_1,\ldots,x_m). \end{aligned}$$

Therefore Y is a group of operators on $C^m(X,G)$.

Consider the co-chain groups

$$C^m\{X, C^n(Y, G)\}, \qquad C^n\{Y, C^m(X, G)\} \tag{2.5}$$

and the co-boundary operators

$$\delta_x : C^m\{X, C^n(Y, G)\} \to C^{m+1}\{X, C^n(Y, G)\},$$
$$\delta_y : C^n\{Y, C^m(X, G)\} \to C^{n+1}\{Y, C^m(X, G)\},$$

which are defined by (1.2) in terms of (2.3) and (2.4). An element in either of the groups (2.5) may be regarded simply as a function of the form

$$c(x_1, \ldots, x_m; y_1, \ldots, y_n).$$

Therefore we identify these groups with each other and denote them both by

$$C^{m,n} = C^{m,n}(X, Y; G).$$

Then δ_x, δ_y become 'partial' co-boundary operators,

$$\delta_x : C^{m,n} \to C^{m+1,n}, \qquad \delta_y : C^{m,n} \to C^{m,n+1}$$

such that $\delta_x \delta_x = 0$, $\delta_y \delta_y = 0$. I say that $\delta_x \delta_y = \delta_y \delta_x$. For it follows from (2.1) that

$$x_1(x_2 \ldots x_{m+1} y \,.\, g) = x_1 \ldots x_{m+1} y \,.\, x_1 g.$$

Therefore, taking $m = n = 1$, we have

$$(\delta_y c)(x; y_1, y_2) = x y_1 \,.\, c(x; y_2) - c(x; y_1 y_2) + c(x; y_1),$$

and $(\delta_x \delta_y c)(x_1, x_2; y_1, y_2)$ is

$$\begin{aligned} & x_1\{x_2 y_1 \,.\, c(x_2; y_2)\} - x_1 c(x_2; y_1 y_2) + x_1 c(x_2; y_1) - \\ & - x_1 x_2 y_1 \,.\, c(x_1 x_2; y_2) + c(x_1 x_2; y_1 y_2) - c(x_1 x_2; y_1) + \\ & + x_1 x_2 y_1 \,.\, c(x_1; x_2 y_2) - c(x_1; x_2 (y_1 y_2)) + c(x_1; x_2 y_1) \\ = {} & x_1 x_2 y_1 \,.\, (\delta_x c)(x_1, x_2; y_2) - (\delta_x c)(x_1, x_2; y_1 y_2) + (\delta_x c)(x_1, x_2; y_1) \\ = {} & (\delta_y \delta_x c)(x_1, x_2; y_1, y_2). \end{aligned}$$

Therefore $\delta_x \delta_y = \delta_y \delta_x$ in the case $m = n = 1$, and the general case follows from a similar argument.

Let $\mathfrak{C}^n = \mathfrak{C}^n(X, Y; G)$ be the group, with vector addition, in which the elements are all ordered sets $(c_0, \ldots, c_n)$, where

$$c_k \in C^{k,n-k}(X, Y; G).$$

I call $(c_0, \ldots, c_n)$ a *vector co-chain*. A homomorphism, $\Delta : \mathfrak{C}^n \to \mathfrak{C}^{n+1}$ is defined by

$$\Delta(c_0, \ldots, c_n) = (c'_0, \ldots, c'_{n+1}),$$

where $$c'_k = \delta_x c_{k-1} + (-1)^k \delta_y c_k \quad (c_{-1} = c_{n+1} = 0).$$

Since $\delta_x \delta_x = 0$, $\delta_y \delta_y = 0$ and $\delta_x \delta_y = \delta_y \delta_x$, it follows that $\Delta\Delta = 0$.

Let $0 \leqslant p \leqslant n$ and let $\mathfrak{C}^n_p \subset \mathfrak{C}^n$ be the sub-group which consists of

the vector co-chains $(c_0,\dots,c_n)$ such that $c_{p+1}=0,\dots,c_n=0$. Obviously $\Delta\mathfrak{C}^{n-1}_{p-1}\subset\mathfrak{C}^n_p$ and we define

$$\mathfrak{H}^n_p=\mathfrak{Z}^n_p-\Delta\mathfrak{C}^{n-1}_{p-1},$$

where $\mathfrak{Z}^n_p=\mathfrak{C}^n_p\cap\Delta^{-1}(0)$. I shall describe an element in $\mathfrak{Z}^n_p$ as a *vector co-cycle* and $\mathfrak{z},\mathfrak{z}'\in\mathfrak{Z}^n_p$ will be called *cohomologous*,† written $\mathfrak{z}\sim\mathfrak{z}'$, if and only if $\mathfrak{z}-\mathfrak{z}'\in\Delta\mathfrak{C}^{n-1}_{p-1}$. I shall denote an element of $\mathfrak{C}^n_p$ by $(c_0,\dots,c_p)^n$.

3. Normalized co-chains

A co-chain $c\in C^{m,n}$ will be called *x-normalized* if and only if either $m=0$ or $m>0$ and the value of c is zero when one of its arguments in X takes the value 1. I define a *y-normalized* co-chain in a similar way and c will be called *(x, y)-normalized* if and only if it is normalized in both x and y. If c is w-normalized ($w=x$ or y) so, obviously, are $\delta_x c$ and $\delta_y c$.

Let $\delta_y c$ be y-normalized, where $c\in C^{m,1}$. Then

$$c(x_1,\dots,x_m;1)=(\delta_y c)(x_1,\dots,x_m;1,1)=0. \tag{3.1}$$

Therefore c is y-normalized. Similarly $c\in C^{1,n}$ is x-normalized if $\delta_x c$ is x-normalized.

A vector co-chain $(c_0,\dots,c_p)^n\in\mathfrak{C}^n_p$ will be called *w-normalized* ($w=x$ or y) if and only if the same is true of c_k for each $k=0,\dots,p$. If $(c_0,\dots,c_p)^n$ is w-normalized so, obviously, is $\Delta(c_0,\dots,c_p)^n$. Let $p=n-1$ or n and let $\Delta(c_0,\dots,c_p)^n$ be (x,y)-normalized. Then I say that there is a vector co-chain $(c^*_0,\dots,c^*_{p-1})^{n-1}$ such that

$$(c_0,\dots,c_p)^n+\Delta(c^*_0,\dots,c^*_{p-1})^{n-1} \tag{3.2}$$

is (x,y)-normalized. For let $0\leqslant i\leqslant p$ and assume that $c_{i+1},\dots,c_p$ are all x-normalized, this condition being vacuous if $i=p$. If $i=0$, then c_i is also x-normalized. Let $i>0$. Since $\delta_y c_{i+1}$ and $\delta_x c_i+(-1)^{i+1}\delta_y c_{i+1}$ are both x-normalized, so is $\delta_x c_i$. Therefore, by treating c_i as an element of $C^i(X,A)$, where $A=C^{n-i}(Y,G)$, it follows from‡ (6.4) in (**1**) that there is a co-chain

$$c^0\in C^{i-1}(X,A)=C^{i-1,n-i}(X,Y;G)$$

such that $c_i+\delta_x c^0$ is x-normalized. Let

$$c^0_{i-1}=c^0,\qquad c^0_j=0\quad\text{if}\quad j\neq i-1\quad(j=0,\dots,p-1).$$

Then $c'_i,\dots,c'_p$ are all x-normalized, where

$$(c'_0,\dots,c'_p)^n=(c_0,\dots,c_p)^n+\Delta(c^0_0,\dots,c^0_{p-1})^{n-1}.$$

† In general $\Delta\mathfrak{C}^{n-1}_{q-1}\cap\mathfrak{C}^n_p\neq\Delta\mathfrak{C}^{n-1}_{p-1}$ if $q>p$. Therefore the relation $\mathfrak{z}\sim\mathfrak{z}'$ depends on the value of p. In the subsequent applications p will have the fixed value $n-1$ and there will be no ambiguity.

‡ The right-hand operators on A do not disturb the argument in (**1**).

Therefore it follows inductively that we may assume $(c_0,...,c_p)^n$ to be x-normalized.

Let $(c_0,...,c_p)^n$ be x-normalized and assume that $c_0,...,c_{k-1}$ are also y-normalized, where $0 \leqslant k \leqslant p$. Since $\delta_x c_{k-1}$ and $\delta_x c_{k-1}+(-1)^k\delta_y c_k$ are both y-normalized, so is $\delta_y c_k$. Since $p = n-1$ or n, it follows from (3.1), if $p = n-1$, or from the definition, if $p = n$, that c_k is also y-normalized if $k = p$. Let $k < p$ and let $N^{r,s} \subset C^{r,s}$ be the sub-group which consists of the x-normalized co-chains and which may be identified with $C^s(Y, N^r)$, where $N^r \subset C^r(X, G)$ is the group of normalized co-chains. Then it follows from (6.4) in (**1**) that there is an element $c' \in N^{k,n-k-1}$ such that $c_k+(-1)^k\delta_y c'$ is y-normalized. Let

$$c'_k = c', \qquad c'_j = 0 \quad \text{if} \quad j \neq k \quad (j = 0,...,p-1).$$

Then $c''_0,...,c''_k$ are all y-normalized, where

$$(c''_0,...,c''_p)^n = (c_0,...,c_p)^n+\Delta(c'_0,...,c'_{p-1})^{n-1}.$$

Also $(c'_0,...,c'_{p-1})^{n-1}$ and hence $\Delta(c'_0,...,c'_{p-1})^{n-1}$ and $(c''_0,...,c''_p)^n$ are x-normalized. Therefore the assertion follows by induction on k.

Hence it follows, as in § 6 of (**1**), that, if $p = n-1$ or n,

(*a*) *every vector co-cycle in* $\mathfrak{Z}^n_p$ *is cohomologous to one which is* (x, y)*-normalized,*

(*b*) *if* $\mathfrak{z} \sim 0$, *where* $\mathfrak{z} \in \mathfrak{Z}^n_p$ *and* $\mathfrak{z}$ *is* (x, y)*-normalized, then* $\mathfrak{z} = \Delta\mathfrak{c}$, *where* $\mathfrak{c} \in \mathfrak{C}^{n-1}_{p-1}$ *is* (x, y)*-normalized.*

Therefore, in defining $\mathfrak{H}^n_p$ ($p = n-1$ or n) we may confine ourselves entirely to co-chains which are (x, y)-normalized and $\mathfrak{C}^n_p$ will now denote the group of (x, y)-normalized co-chains. Notice that an element $(c_0, c_1)^2 \in \mathfrak{Z}^2_1$ is (x, y)-normalized provided only that c_0 is y-normalized. For $c_1 \in C^{1,1}$ is x-normalized because $\delta_x c_1 = 0$. If c_0 and hence $\delta_x c_0$ are y-normalized, so is c_1 because $\delta_y c_1 = \delta_x c_0$.

4. Application to (E, ϕ)

Let (E, ϕ) be an X-extension of Y by G. I write E additively, though it need not be Abelian, and I pick a representative, $e(y) \in \phi^{-1}(y)$, of each $y \in Y$, taking† $e(1) = 0$. Then

$$y.g = e(y)+g-e(y).$$

Since ϕ is an operator homomorphism, we have

$$\phi\{xe(y)\} = x\phi e(y) = xy = \phi e(xy).$$

Therefore

$$xe(y) = c_1(x; y)+e(xy),$$

† It is to be understood that all our sets of representatives satisfy this condition.

where $c_1(x;y) \in G$. Since $x_1\{x_2 e(y)\} = (x_1 x_2)e(y)$, it follows from a familiar argument that $\delta_x c_1 = 0$. We also have

$$\begin{aligned} x(y.g) &= xe(y)+xg-xe(y) \\ &= c_1(x;y)+e(xy)+xg-e(xy)-c_1(x;y) \\ &= c_1(x;y)+xy.xg-c_1(x;y) \\ &= xy.xg, \end{aligned}$$

which is (2.1).

By the ordinary theory of group extensions we have

$$e(y_1)+e(y_2) = c_0(y_1,y_2)+e(y_1y_2),$$

where $c_0 \in C^{0,2}$ and $\delta_y c_0 = 0$. Since

$$\begin{aligned} &xc_0(y_1,y_2)+c_1(x;y_1y_2)+e(x(y_1y_2)) \\ &\quad = x\{c_0(y_1,y_2)+e(y_1y_2)\} \\ &\quad = x\{e(y_1)+e(y_2)\} \\ &\quad = xe(y_1)+xe(y_2) \\ &\quad = c_1(x;y_1)+e(xy_1)+c_1(x;y_2)+e(xy_2) \\ &\quad = c_1(x;y_1)+xy_1.c_1(x;y_2)+e(xy_1)+e(xy_2) \\ &\quad = c_1(x;y_1)+xy_1.c_1(x;y_2)+c_0(xy_1,xy_2)+e(x(y_1y_2)), \end{aligned}$$

we have $\delta_x c_0 = \delta_y c_1$. Therefore $(c_0,c_1)^2 \in \mathfrak{Z}_1^2$. Since $e(1) = 0$, it is easily verified that $(c_0,c_1)^2$ is (x,y)-normalized.

Let $(c_0',c_1')^2$ be defined in the same way as $(c_0,c_1)^2$ by means of a set of representatives $\{e'(y)\}$. Then

$$e'(y) = c(y)+e(y),$$

where $c \in C^{0,1}$. An argument of a familiar sort shows that

$$c_0' = c_0+\delta_y c, \qquad c_1' = c_1+\delta_x c.$$

Therefore $(c_0,c_1)^2 \sim (c_0',c_1')^2$ and the corresponding element of $\mathfrak{H}_1^2$ is uniquely determined by (E,ϕ). Moreover this argument may be reversed to show that a given (x,y)-normalized vector co-cycle in $\mathfrak{Z}_1^2$ which is cohomologous to $(c_0,c_1)^2$, is the one determined by some set of representatives $\{e'(y)\}$.

Let (E',ϕ') be an X-extension of Y by G, which is equivalent to (E,ϕ) under an operator isomorphism $\psi: E \approx E'$. Let $\{e(y)\}$ be a set of representatives in E and let $e'(y) = \psi e(y)$. Then $\{e'(y)\}$ is a set of representatives in E'. Since $\psi x = x\psi$ and $\psi g = g$, it follows that the same vector co-cycle is determined by $\{e'(y)\}$ as by $\{e(y)\}$. Therefore (E,ϕ) and (E',ϕ') determine the same element of $\mathfrak{H}_1^2$.

Conversely, let (E,ϕ) and (E',ϕ') determine the same element of $\mathfrak{H}^2_1$. Then suitable sets of representatives, $\{e(y)\}$ and $\{e'(y)\}$, in E and E', determine the same vector co-cycle, and it may be verified that (E,ϕ) and (E',ϕ') are equivalent under $\psi: E \approx E'$, where

$$\psi\{g+e(y)\} = g+e'(y).$$

Therefore (E,ϕ) and (E',ϕ') are equivalent if and only if they determine the same element of $\mathfrak{H}^2_1$.

Finally a normalized element $c_0 \in C^{0,2}$ such that $\delta_y c_0 = 0$ determines an ordinary extension (E,ϕ), which consists of all pairs (g,y) with addition defined by

$$(g_1,y_1)+(g_2,y_2) = \{g_1+y_1.g_2+c_0(y_1,y_2), y_1y_2\}.$$

Let $c_1 \in C^{1,1}$ be such that $\delta_x c_1 = 0$ and $\delta_y c_1 = \delta_x c_0$. I define $x: E \approx E$ by

$$x(g,y) = \{xg+c_1(x,y), xy\}.$$

Then E becomes an X-extension, and the representatives $e(y) = (0,y)$ determine the vector co-cycle $(c_0,c_1)^2$.

5. (X, Y)-kernels

Let (K,θ) be a Y-kernel, as defined in (**2**), with centre G. We write K additively. Let $\alpha(y)$, with $\alpha(1) = 1$, be an automorphism in the class $\theta(y)$, for each $y \in Y$. I shall describe (K,θ) as an (X,Y)-*kernel* if and only if K, as well as Y, admits X as a group of operators, subject to the condition

$$x\alpha(y) = C[k_1(x;y)]\alpha(xy)x, \tag{5.1}$$

where $$k_1(x;y) \in K, \qquad k_1(1;y) = k_1(x;1) = 0$$

and $C[k_1]$ is the inner automorphism $k \to k_1+k-k_1$. The centre G admits X and also Y as groups of operators, such that $y.g = \alpha(y)g$, and it follows from (5.1) that (2.1) is satisfied. We also have

$$\alpha(y_1)\alpha(y_2) = C[k_0(y_1,y_2)]\alpha(y_1,y_2), \tag{5.2}$$

as in (**2**), where $k_0(y_1,y_2) \in K$. Let

$$\begin{aligned} c_0(y_1,y_2,y_3) &= \alpha(y_1)k_0(y_2,y_3)+k_0(y_1,y_2y_3)-k_0(y_1y_2,y_3)-k_0(y_1,y_2), \\ c_1(x;y_1,y_2) &= xk_0(y_1,y_2)+k_1(x;y_1y_2)-k_0(xy_1,xy_2)- \\ &\qquad -\alpha(xy_1)k_1(x;y_2)-k_1(x;y_1), \\ c_2(x_1,x_2;y) &= x_1k_1(x_2;y)+k_1(x_1;x_2y)-k_1(x_1x_2;y). \end{aligned} \tag{5.3}$$

It may be verified that the left-hand side of each of these equations is an element in G; also that $(c_0, c_1, c_2)^3 \in \mathfrak{Z}_2^3$ and that the corresponding element of $\mathfrak{H}_2^3$ is uniquely determined by the (X, Y)-kernel (K, θ). I shall not prove these facts by direct methods but shall show how they may be deduced from (4).

6. Lyndon's normalization

Let W be the (multiplicative) extension of X by Y, whose elements are all pairs (y, x), with

$$(y_1, x_1)(y_2, x_2) = (y_1 x_1 y_2, x_1 x_2) \quad (x_i \in X,\ y_i \in Y).$$

I identify y with $(y, 1)$ and $\bar{x}$ will denote $(1, x)$. Then $(y, x) = y\bar{x}$, and w, w_i will always denote $y\bar{x}$, $y_i\bar{x}_i$. Notice that $\bar{x}y\bar{x}^{-1} = xy$. Let

$$wg = y.xg \quad (g \in G).$$

Then $wg = g$ if $w = 1$, and it follows from (2.1) that

$$w_1(w_2 g) = y_1.\{x_1(y_2.x_2 g)\} = y_1 x_1 y_2.x_1 x_2 g$$
$$= (w_1 w_2)g.$$

Therefore G admits W as a group of operators according to this rule. I shall use the latter to define δf, where $f \in C^n(W, G)$.

Let $C^n = C^n(W, G)$ and, to simplify the notation, let C^n denote the group of w-normalized co-chains. Let $f \in C^n$, let $0 \leqslant k \leqslant n$, and let $f_k \in C^n$ be defined by

$$f_k(w_1, \ldots, w_n) = y_1' \ldots y_k'.f(x_1, \ldots, x_k, y_{k+1}'', \ldots, y_n''), \tag{6.1}$$

where $y_i' = (x_1 \ldots x_{i-1})y_i$, $y_j'' = (x_{k+1} \ldots x_{j-1})y_j$, with $y_1' = y_1$, $y_{k+1}'' = y_{k+1}$. I describe f as *L-normalized* if and only if $f = f_0 + \ldots + f_n$. Lyndon shows, assuming all the operators to be simple, that the statement analogous to the one containing (3.2), and hence also (a) and (b), are true concerning L-normalization. His arguments can be adapted† to show that the same is true in the case considered here.

Let $C_L^n \subset C^n$ be the sub-group consisting of the (w-normalized) co-chains which are also L-normalized. A homomorphism $\rho : C_L^n \to \mathfrak{C}^n$ is given by $\rho f = (c_0, \ldots, c_n)$, where $f \in C_L^n$ and

$$c_k(x_1, \ldots, x_k; y_{k+1}, \ldots, y_n) = f(x_1, \ldots, x_k, y_{k+1}, \ldots, y_n). \tag{6.2}$$

Obviously $\rho : C_L^n \approx \mathfrak{C}^n$, the inverse ρ^{-1} being given by (6.1) and (6.2). It follows from arguments similar to those in § 4 of (4) that $\delta C_L^n \subset C_L^{n+1}$ and that $\rho\delta = \Delta\rho$.

† Lyndon's A, B are my X, Y. His b_j becomes $x_i \ldots x_{j-1}\, y_j$, the value of i depending on the context, and p, in his (5.1), becomes $y_1' \ldots y_{h-1}'\, p$ $(y_j' = x_1 \ldots x_{j-1}\, y_j)$.

Let $\lambda: X \to W$ be the homomorphism which is given by $\lambda x = \bar{x}$, let $\lambda^*: C^n \to C^n(X, G)$ be the homomorphism induced by λ, and let $C^n_* = \lambda^{*-1}(0)$. Since $\bar{x}g = xg$, it follows that $\lambda^*\delta = \delta\lambda^*$, whence $\delta C^n_* \subset C^{n+1}_*$. Let $\delta_*: C^n_* \to C^{n+1}_*$ be the homomorphism induced by δ and let $H^0_*, H^1_*, \ldots$ be the cohomology groups which are defined in terms of $C^0_*, C^1_*, \ldots$ and the operator δ_*. Let $f \in C^n_*$ and let δf be L-normalized. Then Lyndon's argument shows that $f + \delta_* g$ is L-normalized for some $g \in C^{n-1}_*$. Thus H^n_* may be defined in terms of the groups $C^n_* \cap C^n_L$. Since $f \in C^n_*$ if and only if $f(\bar{x}_1, \ldots, \bar{x}_n) = 0$, it follows that

$$\rho(C^n_* \cap C^n_L) = \mathfrak{C}^n_{n-1}.$$

Therefore ρ induces an isomorphism $H^n_* \approx \mathfrak{H}^n_{n-1}$ for each $n = 0, 1, \ldots$.

Now let (K, θ), $\alpha(y)$, etc., mean the same as in § 5. We define a W-kernel $(K, \bar{\theta})$ by taking $\bar{\theta}(w) = \theta(y)\gamma(x)$, where $\gamma(x)$ is the class of the automorphism $x: K \approx K$. We take $\bar{\alpha}(w) = \alpha(y)x$ as the representative of $\bar{\theta}(w)$. It follows from (5.1) and (5.2) that

$$\begin{aligned}\bar{\alpha}(w_1)\bar{\alpha}(w_2) &= \alpha(y_1)x_1\,\alpha(y_2)x_2 \\ &= \alpha(y_1)C[k_1(x_1; y_2)]\alpha(x_1 y_2)x_1 x_2 \\ &= C[k(w_1, w_2)]\bar{\alpha}(w_1 w_2),\end{aligned}$$

where $k(w_1, w_2) = \alpha(y_1)k_1(x_1; y_2) + k_0(y_1, x_1 y_2)$. Let

$$g(w_1, w_2, w_3) = \bar{\alpha}(w_1)k(w_2, w_3) + k(w_1, w_2 w_3) - \\ - k(w_1 w_2, w_3) - k(w_1, w_2). \qquad (6.3)$$

Since $k(\bar{x}_1, \bar{x}_2) = 0$, it follows that the co-cycle g belongs to C^3_*. Therefore $(K, \bar{\theta})$, restricted by the conditions $\bar{\theta}(\bar{x}) = \gamma(x)$ and $k(\bar{x}_1, \bar{x}_2) = 0$, determines a unique element of H^3_*. It follows from (6.3) and (5.3) that $g \in C^3_* \cap C^3_L$ and that $\rho g = (c_0, c_1, c_2)^3$.

If two W-kernels $(K_1, \bar{\theta}_1)$, $(K_2, \bar{\theta}_2)$ are related in this way to (X, Y)-kernels (K_1, θ_1), (K_2, θ_2), then their G-product† is obviously related in the same way to the G-product of (K_1, θ_1), (K_2, θ_2).

Let $(K, \bar{\theta})$ be a given W-kernel, with representatives $\bar{\alpha}(w) \in \bar{\theta}(w)$ such that

$$\bar{\alpha}(\bar{x}_1)\bar{\alpha}(\bar{x}_2) = \bar{\alpha}(\bar{x}_1 \bar{x}_2). \qquad (6.4)$$

Then K admits X as a group of operators, with $xk = \bar{\alpha}(\bar{x})k$. Let

$$\bar{\alpha}(w_1)\bar{\alpha}(w_2) = C[k(w_1, w_2)]\bar{\alpha}(w_1 w_2),$$

let $k(w) = k(y, \bar{x})$, and let

$$\bar{\alpha}_1(\bar{w}) = C[k(w)]\bar{\alpha}(w) = \bar{\alpha}(y)\bar{\alpha}(\bar{x}) = \bar{\alpha}_1(y)x.$$

† See (4.1) in (**2**). Obviously the G-product of (K_1, θ_1), (K_2, θ_2), with X operating in the natural way, is an (X, Y)-kernel.

The relation (5.1) is satisfied with

$$\alpha(y) = \bar{\alpha}_1(y); \qquad k_1(x; y) = k(\bar{x}, y).$$

Therefore (K, θ), with $\theta = \bar{\theta}|Y$, is an (X, Y)-kernel which is in the above relationship to $(K, \bar{\theta})$.

Let g be a given co-cycle in C^3_* and let $(K, \bar{\theta})$ be the W-kernel which is defined in terms of g by the construction in § 9 of (**2**), with the following modifications. The generators of the (additive) free group F shall be all pairs $[w_1, y_2]$, with $y_2 \in Y$, not all pairs $[w_1, w_2]$. The function h shall be given by $h(w_1, w_2) = [w_1, y_2]$, with $h(w_1, \bar{x}_2) = h(1, w_2) = 0$. Then $\beta(\bar{x}) = \bar{\alpha}(\bar{x})$ satisfies (6.4), and $(K, \bar{\theta})$ determines an (X, Y)-kernel whose invariant is the element of H^3_* which corresponds to g.

Therefore the theory of (X, Y)-kernels, expressed in terms of H^n_* or $\mathfrak{H}^n_{n-1}$ $(n = 2, 3)$, is entirely analogous to the theory of Y-kernels which is developed in (**2**). In particular, if (K, θ) has an X-extension, the equivalence classes of its X-extensions are in (1, 1) correspondence with the elements of $\mathfrak{H}^2_1$. On taking $K = G$ this leads to an alternative treatment of X-extensions of Y by G.

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DIFFRACTION BY A METAL WEDGE AT LARGE ANGLES

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1. Boundary conditions

By a metal we mean a homogeneous conducting dielectric with a complex refractive index

$$n = n_0 - i\nu, \tag{1}$$

where ν is large but not infinite. This condition is best fulfilled at the frequency of sodium light by aluminium and platinum, for which ν is 5·25 and 4·25 respectively. The general theoretical problem of diffraction is that of electric waves falling on a mass of dielectric bounded by a closed surface S. We have to find solutions of the field equations inside and outside S, with the tangential components of electric and magnetic intensity continuous in passing through S. This problem is intractable, but more progress has been made with a perfect conductor (ν infinite), where we have only to solve the external field equations with the tangential component of electric intensity zero on S. We now show that a similar condition holds for metals, so that here also we need not consider the inside of S.

That some such theory may be possible is suggested by Kelvin's calculation (**2**) of the resistance of a wire of any cross-section at high frequencies. He took the conductivity so great that the thickness of the shell of sensible current was a small fraction of the radius of curvature at all but a few isolated points. Metals have a similar skin effect at optical frequencies, to which they owe their great opacity. We shall use the metre-kilogram-second system of electrical units, in which the velocity of light in free space is $c = (1/\epsilon_0\mu_0)^{\frac{1}{2}} = 2{\cdot}998\times10^8$ m./sec. Only periodic phenomena are considered, with a time-factor $\exp i\omega t$. A plane electric wave in free space is represented by $\exp(i\omega t - ikx)$, and in the metal by $\exp(i\omega t - inkx)$, where

$$k = \omega/c = 2\pi/\lambda \tag{2}$$

and λ is the wave-length in metres. The small thickness of the surface layer appears to justify the assumption that the electric and magnetic

intensities at a point (x, y, z) in the metal near the origin of coordinates are given approximately by

$$(E_x, E_y) = (E_{x0}, E_{y0})\exp(i\omega t+inkz), \qquad E_z = E_{z0}\exp(i\omega t+inkz)/n^2,$$
$$(H_x, H_y, H_z) = (H_{x0}, H_{y0}, H_{z0})\exp(i\omega t+inkz),$$

where the origin is taken in the surface S, the axes of x and y tangential to S, and z normal and away from S. In these formulae E_{x0}, H_{x0},... are the amplitudes of electric and magnetic intensity just outside the surface, and are functions of x and y. Four of the field equations are

$$\partial H_{z0}/\partial y - inkH_{y0} = i\omega\epsilon_0 n^2 E_{x0}, \qquad inkH_{x0} - \partial H_{z0}/\partial x = i\omega\epsilon_0 n^2 E_{y0},$$
$$(\partial E_{z0}/\partial y)/n^2 - inkE_{y0} = -i\omega\mu_0 H_{x0}, \qquad inkE_{x0} - (\partial E_{z0}/\partial x)/n^2 = -i\omega\mu_0 H_{y0},$$

which yield the conditions

$$E_{x0} = -(\mu_0/\epsilon_0)^{\frac{1}{2}} H_{y0}/n, \qquad E_{y0} = (\mu_0/\epsilon_0)^{\frac{1}{2}} H_{x0}/n$$

when n is large.

A check is afforded by the formulae of Hoerschelmann (**1**), who found very general solutions of the field equations for a dielectric with a plane face. These solutions are reproduced in Pidduck's book *Currents in Aerials and High-frequency Networks*, and satisfy the conditions when n is large.

Let S in particular be an infinitely long cylinder. Change the axes so that the axis of z is parallel to the generators. Let the electric and magnetic intensities be independent of z. Two main problems then arise, in which the electric intensity is everywhere parallel to the axis of z (parallel polarization) and perpendicular to it (perpendicular polarization). The magnetic intensity in perpendicular polarization is parallel to the axis of z. The other axes are taken to be the normal and tangent to S at any point. Transferring the field equations outside S to this system of curvilinear coordinates we find that the boundary conditions take the form

$$E_{z0} = -\frac{i}{kn}\left(\frac{\partial E_z}{\partial N}\right)_0, \qquad \left(\frac{\partial H_z}{\partial N}\right)_0 = \frac{ik}{n} H_{z0} \tag{3}$$

for parallel and perpendicular polarizations respectively, where $\partial/\partial N$ denotes differentiation along the outward normal to dS. If the diffraction problem has been solved for perfect conduction, the terms containing $1/n$ can be taken as known to this order of approximation.

2. Green's functions for a wedge

Let $P(x, y)$ and $P_0(x_0, y_0)$ be two points in the same cross-section of the cylinder S. Let $G(-)(P, P_0)$ be a solution of the two-dimensional wave equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + k^2 u = 0$$

which vanishes on S and becomes infinite at P_0 in such a way that $G(-) \approx \log(1/PP_0)$ when PP_0 is small. Let $G(+)(P, P_0)$ be a solution of the equation such that $\partial G(+)(P, P_0)/\partial N$ vanishes on S and $G(+)$ becomes infinite at P_0 in the same way. Then the solutions of the equation which have assigned values of u and $\partial u/\partial N$ on S are respectively

$$u = \frac{1}{2\pi}\int u_0 \frac{\partial G(-)(P, P_0)}{\partial N}\, dr_0, \qquad u = -\frac{1}{2\pi}\int \left(\frac{\partial u}{\partial N}\right)_0 G(+)(P, P_0)\, dr_0 \tag{4}$$

where dr_0 is an element of arc of the cross-section, and integration is over the perimeter. The functions $G(-)$ and $G(+)$ are unaltered when P and P_0 are interchanged.

For a wedge, take the edge as axis of z. Let r, θ be the polar coordinates of P and r_0, θ_0 those of P_0. Let the region outside the wedge, in which we have to find the wave motion, lie between the planes $\theta = 0$ (which we call the 'top' surface) and $\theta = m\pi$ (which we call the 'bottom' surface). The coefficient m is 2 for an infinitely thin sheet, but in general is less than 2, for example 1·911 for a wedge of angle 16°. Green's functions for a wedge were found by Macdonald (**3**); see also Schwarzschild (**5**). We have

$$G(\mp)(P, P_0) = G_1(r, \theta, r_0, \theta_0) \mp G_2(r, \theta, r_0, \theta_0), \tag{5}$$

where

$$G_1(r, \theta, r_0, \theta_0) = -\frac{1}{4}\int_A \frac{H_0^{(2)}\{k(r^2+r_0^2-2rr_0\cos m\alpha)^{\frac{1}{2}}\}\sin\alpha\, d\alpha}{\cos\alpha - \cos\{(\theta-\theta_0)/m\}} \tag{6}$$

and G_2 is obtained by writing $\theta+\theta_0$ for $\theta-\theta_0$ in the denominator. $H_0^{(2)}(z)$ is Hankel's function $J_0(z) - iY_0(z)$. The integral is along an arc of a curve A in the complex plane, beginning at the point $-(\pi/m)+i\infty$, passing below the point $\cos m\alpha = (r^2+r_0^2)/2rr_0$ and above the real axis, and ending at the point $(\pi/m)+i\infty$. If it is deformed after the manner of Schwarzschild, i.e. into the straight lines joining $-(\pi/m)+i\infty$, $-(\pi/m)$, (π/m), $(\pi/m)+i\infty$, and if we write

$$\left.\begin{aligned} C &= -\cos(\pi/m), \qquad S = \sin(\pi/m) \\ c &= \cos(\theta/m), \qquad s = \sin(\theta/m) \\ \phi(c, x) &= \frac{C + c\cosh x}{C^2+c^2+2Cc\cosh x+\sinh^2 x} \\ \Phi(c, x) &= \frac{-(C^2+c^2)\cosh x - 2Cc + \cosh x \sinh^2 x}{(C^2+c^2+2Cc\cosh x+\sinh^2 x)^2} \end{aligned}\right\}, \tag{7}$$

$$f(r,\theta,r_0) = \begin{cases} -\tfrac{1}{2}\pi i H_0^{(2)}\{k(r^2+r_0^2-2rr_0\cos\theta)^{\frac{1}{2}}\} & \text{if} \quad 0<\theta<\pi \\ 0 \quad \text{if} \quad \pi<\theta<m\pi, \end{cases} \Bigg\},$$

$$F(r,\theta,r_0) = \begin{cases} -\tfrac{1}{2}\pi i k r\sin\theta \dfrac{H_1^{(2)}\{k(r^2+r_0^2-2rr_0\cos\theta)^{\frac{1}{2}}\}}{(r^2+r_0^2-2rr_0\cos\theta)^{\frac{1}{2}}} & \text{if} \quad 0<\theta<\pi \\ 0 \quad \text{if} \quad \pi<\theta<m\pi \end{cases} \Bigg\}, \tag{8}$$

we have

$$\left.\begin{aligned} G_1(r,\theta,r_0,\theta_0) &= \begin{Bmatrix} f(r,\theta,r_0) \\ f(r,m\pi-\theta,r_0) \end{Bmatrix} + \\ &\quad + \tfrac{1}{2}iS\int_0^\infty H_0^{(2)}\{k(r^2+r_0^2+2rr_0\cosh mx)^{\frac{1}{2}}\}\begin{Bmatrix} \phi(c,x) \\ \phi(-c,x) \end{Bmatrix} dx \\ \frac{\partial G_1(r,\theta,r_0,\theta_0)}{\partial N} &= \begin{Bmatrix} F(r,\theta,r_0) \\ F(r,m\pi-\theta,r_0) \end{Bmatrix} + \\ &\quad + \frac{iSs}{2mr_0}\int_0^\infty H_0^{(2)}\{k(r^2+r_0^2+2rr_0\cosh mx)^{\frac{1}{2}}\}\begin{Bmatrix} \Phi(c,x) \\ \Phi(-c,x) \end{Bmatrix} dx \end{aligned}\right\}, \tag{9}$$

where the top line in each bracket is taken on the top surface, and the bottom line on the bottom surface.* Also $G_2 = G_1$ and

$$\partial G_2/\partial N = -\partial G_1/\partial N$$

on both surfaces of the wedge, so that in equations (4) we can put

$$G(\mp)(P,P_0) = 2G_1(r,\theta,r_0,\theta_0). \tag{10}$$

3. Diffraction of line sources by a perfectly conducting wedge

Sources are taken to be of two types:

(i) A line of electric doublets parallel to the axis of z, through the point $P_1(r_1,\theta_1)$, extending to infinity in both directions. The electric moment is parallel to the axis of z and of magnitude $\exp i\omega t$ per metre at all points. This is a source of parallel polarization;

(ii) A similar line of doublets whose electric moment is parallel to the plane of x, y, and makes an angle β with the radius vector from the edge. This is a source of perpendicular polarization.

Parallel polarization. The electric intensity is expressed in terms of Hertz's vector $\mathbf{\Pi}$, which takes from each element of electric moment $\mathbf{M}$ a contribution $\mathbf{M}\exp(-ikR)/4\pi\epsilon_0 R$ at distance R. For an unimpeded source in parallel polarization, $\mathbf{E}$ and $\mathbf{\Pi}$ are parallel to the edge, and

$$E_z = k^2\Pi_z = -ik^2H_0^{(2)}(kR)\exp i\omega t/4\epsilon_0,$$

* We are taking $m > 1$ and P_0 on the surface of the wedge.

where $R = PP_1$. If R is small,

$$E_z \approx k^2 \log(1/kR) \exp i\omega t/2\pi\epsilon_0.$$

When the wedge is present, all the conditions of the problem are satisfied by

$$E_z = k^2 G(-)(P, P_1) \exp i\omega t/2\pi\epsilon_0, \tag{11}$$

since E_z is zero on the wedge and becomes infinite at P_1 in the required order. To evaluate $G_1(P, P_1)$ when P_1 is not on the wedge, replace θ in (8) and (9) by $\theta - \theta_1$. To evaluate $G_2(P, P_1)$, replace θ by $\theta + \theta_1$.

Let P in particular be at a large distance from the edge, P_1 remaining at a finite distance. Then $H_0^{(2)}(z)$ can be put approximately equal to $(1+i)\exp(-iz)/(\pi z)^{\frac{1}{2}}$ and $\{r^2 + r_1^2 - 2rr_1\cos(\theta-\theta_1)\}^{\frac{1}{2}}$ to $r - r_1\cos(\theta-\theta_1)$. The electric intensity is the sum of four terms, two $H_0^{(2)}$ terms, and two integrals. The main part of the integrals comes from the immediate neighbourhood of $x = 0$, where $\cosh mx$ is stationary, in accordance with Kelvin's principle of stationary phase. Replacing $\cosh mx$ by $1 + \frac{1}{2}m^2x^2$ as in Kelvin's method, the integrals become approximately

$$\frac{(1+i)\exp(-ikr - ikr_1)}{(\pi kr)^{\frac{1}{2}}(C+c')} \int_0^\infty \exp(-\tfrac{1}{2}ikr_1 m^2 x^2)\, dx = \frac{\exp(-ikr - ikr_1)}{mk(rr_1)^{\frac{1}{2}}(C+c')},$$

where $c' = \cos\{(\theta \mp \theta_1)/m\}$. The electric intensity at a distant point is therefore

$$\begin{aligned} E_z \approx\ & -i\frac{(1+i)k^2\exp(i\omega t - ikr)}{4\epsilon_0(\pi kr)^{\frac{1}{2}}} \times \\ & \times[\exp\{ikr_1\cos(\theta-\theta_1)\} - \exp\{ikr_1\cos(\theta+\theta_1)\}] + \\ & + \frac{iSk\exp(i\omega t - ikr - ikr_1)}{4\pi\epsilon_0 m(rr_1)^{\frac{1}{2}}}\left[\frac{1}{C+\cos\{(\theta-\theta_1)/m\}} - \frac{1}{C+\cos\{(\theta+\theta_1)/m\}}\right] \end{aligned} \tag{12}$$

where the terms in the first [ ] are present only when $|\theta - \theta_1| < \pi$ and $|\theta + \theta_1| < \pi$ respectively.

Perpendicular polarization. Here

$$(\mu_0/\epsilon_0)^{\frac{1}{2}} H_z = ik(\partial\Pi_y/\partial x - \partial\Pi_x/\partial y).$$

For the unimpeded line source

$$(\mu_0/\epsilon_0)^{\frac{1}{2}} H_z = \frac{k}{4\epsilon_0}\left[\cos(\beta+\theta_1)\frac{\partial}{\partial y_1} - \sin(\beta+\theta_1)\frac{\partial}{\partial x_1}\right] H_0^{(2)}(kR)\exp i\omega t,$$

where x_1, y_1 are the Cartesian coordinates of P_1. For the line source with wedge

$$(\mu_0/\epsilon_0)^{\frac{1}{2}} H_z = -\frac{ik}{2\pi\epsilon_0}\left[\sin\beta\frac{\partial}{\partial r_1} - \frac{\cos\beta}{r_1}\frac{\partial}{\partial\theta_1}\right] G(+)(P, P_1)\exp i\omega t. \tag{13}$$

The magnetic intensity at a distant point is given by

$$(\mu_0/\epsilon_0)^{\frac{1}{2}}H_z \approx -\frac{(1+i)k\exp(i\omega t-ikr)}{4\epsilon_0(\pi kr)^{\frac{1}{2}}}\left[\sin\beta\frac{\partial}{\partial r_1}-\frac{\cos\beta}{r_1}\frac{\partial}{\partial\theta_1}\right]\times$$
$$\times[\exp\{ikr_1\cos(\theta-\theta_1)\}+\exp\{ikr_1\cos(\theta+\theta_1)\}]+$$
$$+\frac{Sk\exp(i\omega t-ikr)}{4\pi\epsilon_0 m(kr)^{\frac{1}{2}}}\left[\sin\beta\frac{\partial}{\partial r_1}-\frac{\cos\beta}{r_1}\frac{\partial}{\partial\theta_1}\right]\times$$
$$\times\frac{\exp(-ikr_1)}{(kr_1)^{\frac{1}{2}}}\left[\frac{1}{C+\cos\{(\theta-\theta_1)/m\}}+\frac{1}{C+\cos\{(\theta+\theta_1)/m\}}\right]. \quad (14)$$

4. Correction for a metal wedge

Parallel polarization. The electric intensity E_z is no longer zero on the wedge, but equal to $-i(\partial E_z/\partial N)_0/nk$ from (3). To order $1/n$, $(\partial E_z/\partial N)_0$ can be put equal to the values obtained from (10) and (11), that is

$$\frac{k^2}{\pi\epsilon_0}\frac{\partial G_1(P_1,P_0)}{\partial N}\exp i\omega t.$$

The correction to E_z is therefore from (4)

$$\Delta E_z = -\frac{ik\exp i\omega t}{\pi^2\epsilon_0}\int\frac{\partial G_1(P,P_0)}{\partial N}\frac{\partial G_1(P_1,P_0)}{\partial N}\frac{dr_0}{n}, \quad (15)$$

where n need not be the same at all points.

Perpendicular polarization. The correction to H_z is similarly, from (4) and (13),

$$(\mu_0/\epsilon_0)^{\frac{1}{2}}\Delta H_z = -\frac{k^2\exp i\omega t}{\pi^2\epsilon_0}\int G_1(P,P_0)\left[\sin\beta\frac{\partial}{\partial r_1}-\frac{\cos\beta}{r_1}\frac{\partial}{\partial\theta_1}\right]G_1(P_1,P_0)\frac{dr_0}{n}. \quad (16)$$

Macdonald found Green's functions first as series, from which we see that $G_1(P,P_0)$ is constant $\times\{1+O(r_0^{1/m})\}$ when r_0 is small, and $\partial G_1(P,P_0)/\partial N$ of order $r_0^{1/m-1}$. When r_0/r is large, $G_1(P,P_0)$ is of order $\exp(-ikr_0)/r_0^{\frac{1}{2}}$ and $\partial G_1(P,P_0)/\partial N$ of order $\exp(-ikr_0)/r_0^{\frac{1}{2}}$. The integrals (15) and (16) therefore converge in general. Even if the angle of the wedge is infinitely small, i.e. $m=2$, the integrals converge since the terms involving $1/r_0$ cancel in the integration along the perimeter.

5. Parallel light diffracted at large distances

If r is large,

$$G_1(r,\theta,r_0,\theta_0) \approx \frac{(1+i)\exp(-ikr)}{(\pi kr)^{\frac{1}{2}}}G_1(\theta,r_0,\theta_0), \quad (17)$$

where we replace $f(r,\theta,r_0)$ and $F(r,\theta,r_0)$ in (8) by the functions

$$\left.\begin{aligned} f(\theta,r_0) &= \begin{cases} -\tfrac{1}{2}\pi i\exp(ikr_0\cos\theta) & \text{if } 0<\theta<\pi, \\ 0 & \text{if } \pi<\theta<m\pi \end{cases} \\ F(\theta,r_0) &= \begin{cases} \tfrac{1}{2}\pi k\sin\theta\exp(ikr_0\cos\theta) & \text{if } 0<\theta<\pi, \\ 0 & \text{if } \pi<\theta<m\pi \end{cases} \end{aligned}\right\}, \tag{18}$$

and have

$$\left.\begin{aligned} &G_1(\theta,r_0,\theta_0) \\ &\quad=\begin{Bmatrix} f(\theta,r_0) \\ f(m\pi-\theta,r_0) \end{Bmatrix}+\tfrac{1}{2}iS\int_0^\infty \exp(-ikr_0\cosh mx)\begin{Bmatrix}\phi(c,x)\\ \phi(-c,x)\end{Bmatrix}dx \\ &\frac{\partial G_1(\theta,r_0,\theta_0)}{\partial N} \\ &\quad=\begin{Bmatrix} F(\theta,r_0) \\ F(m\pi-\theta,r_0) \end{Bmatrix}+\frac{iSs}{2mr_0}\int_0^\infty \exp(-ikr_0\cosh mx)\begin{Bmatrix}\Phi(c,x)\\ \Phi(-c,x)\end{Bmatrix}dx \end{aligned}\right\}. \tag{19}$$

Parallel polarization. If P_1 is a large distance from the edge, the electric intensity at the edge of the unimpeded line source is approximately

$$-i(1+i)k^2\exp(i\omega t-ikr_1)/4\epsilon_0(\pi kr_1)^{\frac{1}{2}}.$$

Multiplying by $4\epsilon_0(\pi kr_1)^{\frac{1}{2}}/-i(1+i)k^2\exp(-ikr_1)$ and making r_1 infinite, we have the field in parallel light such that the incident electric intensity at the edge is $\exp i\omega t$. The first line of (12) must be modified since r_1 was there taken to be finite and r/r_1 large, whereas r/r_1 must now be small though r may be large. The field of a wedge of perfect conductivity is therefore

$$E_z \approx \exp i\omega t[\exp\{ikr\cos(\theta-\theta_1)\}-\exp\{ikr\cos(\theta+\theta_1)\}]- \\ -\frac{S\exp(i\omega t-ikr)}{(1+i)m(\pi kr)^{\frac{1}{2}}}\left[\frac{1}{C+\cos\{(\theta-\theta_1)/m\}}-\frac{1}{C+\cos\{(\theta+\theta_1)/m\}}\right] \tag{20}$$

and

$$\Delta E_z=\frac{4(1+i)\exp(i\omega t-ikr)}{\pi^2k(\pi kr)^{\frac{1}{2}}}\int\frac{\partial G_1(\theta,r_0,\theta_0)}{\partial N}\frac{\partial G_1(\theta_1,r_0,\theta_0)}{\partial N}\frac{dr_0}{n}. \tag{21}$$

Perpendicular polarization. Similarly, from (13) and (15),

$$(\mu_0/\epsilon_0)^{\frac{1}{2}}H_z \approx \exp i\omega t[\exp\{ikr\cos(\theta-\theta_1)\}+\exp\{ikr\cos(\theta+\theta_1)\}]- \\ -\frac{S\exp(i\omega t-ikr)}{(1+i)m(\pi kr)^{\frac{1}{2}}}\left[\frac{1}{C+\cos\{(\theta-\theta_1)/m\}}+\frac{1}{C+\cos\{(\theta+\theta_1)/m\}}\right] \tag{22}$$

and

$$(\mu_0/\epsilon_0)^{\frac{1}{2}}\Delta H_z=\frac{4(1+i)k\exp(i\omega t-ikr)}{\pi^2(\pi kr)^{\frac{1}{2}}}\int G_1(\theta,r_0,\theta_0)G_1(\theta_1,r_0,\theta_0)\frac{dr_0}{n}. \tag{23}$$

The terms in the first line of (20) and (22) are present only when $|\theta-\theta_1| < \pi$ and $|\theta+\theta_1| < \pi$ respectively. These equations are not new, but were found in principle by Poincaré.

When r_0 is small, Macdonald's series gives

$$G_1(\theta, r_0, \theta_0) \approx -\frac{i\pi}{2m} \mp \frac{i\pi c \exp(i\pi/2m) k^{1/m} r_0^{1/m}}{2^{1/m}\Gamma(1/m)},$$

$$\frac{\partial G_1(\theta, r_0, \theta_0)}{\partial N} \approx -\frac{i\pi s \exp(i\pi/2m) k^{1/m} r_0^{1/m-1}}{2^{1/m} m \Gamma(1/m)}. \tag{24}$$

When r_0 is large, the second terms of G_1 and $\partial G_1/\partial N$ in (19) are approximately

$$\frac{(1+i)\pi^{\frac{1}{2}} S \exp(-ikr_0)}{4m(C\pm c)(kr_0)^{\frac{1}{2}}}, \qquad -\frac{(1+i)\pi^{\frac{1}{2}} Ss \exp(-ikr_0)}{4m^2(C\pm c)^2 k^{\frac{1}{2}} r_0^{\frac{3}{2}}}. \tag{25}$$

The upper sign is taken in cases of ambiguity on the top surface of the wedge, and the lower sign on the bottom surface. The integrals (21) and (23) do not converge unconditionally, since r and r_1 are no longer finite; but they do when light falls on only one surface of the wedge and is observed away from the region of reflection. For, if

$$0 < \theta_1 < m\pi - \pi, \qquad \pi < \theta < m\pi, \tag{26}$$

one of the functions $f(\theta, r_0)$, $f(\theta_1, r_0)$ is always zero, and similarly for other pairs which might form a divergent product.

6. Comparison with experiment

For comparison with the experimental results of Savornin (4) we consider the case of normal incidence ($\theta_1 = \frac{1}{2}\pi$), the diffraction being observed in the geometrical shadow. Under these conditions n is constant.

It is easy to prove that an integration by parts, an inversion of the order of integration, and an integration with respect to r_0 lead to the expression

$$\begin{aligned}\Delta E_z = \frac{S\exp(i\omega t - ikr)}{(1+i)m(\pi kr)^{\frac{1}{2}}}\Bigg[&\frac{2ms}{\pi n}\int_0^\infty F_1(c,x)\tanh mx\,dx - \\ &-\frac{2ms_1}{\pi n}\sin(m\pi-\theta)\int_0^\infty \frac{F_1(-c_1,x)\sinh mx}{\cosh mx - \cos(m\pi-\theta)}\,dx - \\ &-\frac{2Sss_1 m}{\pi^2 n}\int_0^\infty\int_0^\infty \{F_1(c,x)F_1(c_1,y) - F_1(-c,x)F_1(-c_1,y)\}\times \\ &\times\frac{\sinh mx \sinh my}{\cosh mx + \cosh my}\,dxdy\Bigg],\end{aligned} \tag{27}$$

where

$$F_1(c,x) = \int \Phi(c,x)\,dx = \frac{-\sinh x}{C^2+c^2+2Cc\cosh x+\sinh^2 x}. \qquad (28)$$

For ΔH_z we omit the integration by parts and obtain

$$(\mu_0/\epsilon_0)^{\frac{1}{2}}\Delta H_z$$
$$= \frac{S\exp(i\omega t-ikr)}{(1+i)m(\pi kr)^{\frac{1}{2}}}\left[\frac{2m}{\pi n}\int_0^\infty \left\{\frac{\phi(-c_1,x)}{\cosh mx-\cos(m\pi-\theta)}-\frac{\phi(c,x)}{\cosh mx}\right\}dx+\right.$$
$$\left.+\frac{2mS}{\pi^2 n}\int_0^\infty\int_0^\infty \frac{\phi(c,x)\phi(c_1,y)-\phi(-c,x)\phi(-c_1,y)}{\cosh mx+\cosh my}\,dxdy\right]. \qquad (29)$$

The integrals are evaluated numerically over the lower part of the range of integration. In the upper part we make use of the formula

$$\int_\gamma^\infty F_1(c,x)\,dx = -\frac{1}{Ss}\tanh^{-1}\frac{Ss}{Cc+\cosh\gamma}$$

and the approximations

$$\cosh mx \sim \tfrac{1}{2}e^{mx}, \qquad \tanh mx \sim 1, \qquad \phi(c,x) \sim 2ce^{-x}.$$

The computation is laborious and we have calculated the values of I_P, I_N, ρ, ψ (as defined by Savornin) at a deviation of 45° ($\theta = 315°$) for a steel wedge of angle 16° in sodium light. In this case

$$m = 86/45 = 1{\cdot}911, \qquad n = 2{\cdot}30-3{\cdot}24i,$$

and we obtain

$$E_z+\Delta E_z = (1{\cdot}33+0{\cdot}102i)\frac{S\exp(i\omega t-ikr)}{(1+i)m(\pi kr)^{\frac{1}{2}}},$$

$$\left(\frac{\mu_0}{\epsilon_0}\right)^{\frac{1}{2}}(H_z+\Delta H_z) = (3{\cdot}29-0{\cdot}773i)\frac{S\exp(i\omega t-ikr)}{(1+i)m(\pi kr)^{\frac{1}{2}}}$$

which give

$$I_P = 1{\cdot}78, \qquad I_N = 11{\cdot}42, \qquad \rho = 24{\cdot}15°, \qquad \psi = 6°.$$

It will be seen that I_N and ψ agree well with experiment whereas I_P, ρ are rather too small and large respectively, a better agreement than might have been expected in view of the smallness of n.

We wish to express our appreciation of the considerable assistance afforded by Mr. D. F. Ferguson in the computation.

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THREE SUMMATIONS DUE TO RAMANUJAN

By H. F. SANDHAM (*Dublin*)

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SOME time ago on my communicating a simple proof of Ramanujan's result

$$\sum_1^\infty \frac{n^{13}}{e^{2\pi n}-1} = \frac{1}{24}$$

to my friend Dr. R. Cooper he replied giving me a very interesting proof of Ramanujan's result

$$\sum_1^\infty \frac{n}{e^{2\pi n}-1} = \frac{1}{24} - \frac{1}{8\pi}.$$

I have added a proof of another result due to Ramanujan:

$$\sum_1^\infty \frac{\coth \pi n}{n^7} = \frac{19\pi^7}{56700}.$$

It will readily be seen that the proofs of the first and third results are easily generalized and that the paper provides elementary means of evaluating

$$\sum_{n=1}^\infty \frac{n^{4p+1}}{e^{2\pi n}-1},$$

when p is a positive, zero, or negative integer.

The arguments are as follows.

1. We have

$$e^{-2\pi x}+e^{-4\pi x}+e^{-6\pi x}+\ldots = \frac{1}{e^{2\pi x}-1}$$

$$= \frac{1}{2\pi x} - \frac{1}{2} + \frac{1}{2\pi}\sum_1^\infty \left(\frac{1}{x+in}+\frac{1}{x-in}\right).$$

Differentiating thirteen times gives

$$1^{13}e^{-2\pi x}+2^{13}e^{-4\pi x}+3^{13}e^{-6\pi x}+\ldots$$

$$= \frac{13!}{(2\pi x)^{14}} + \frac{13!}{(2\pi)^{14}}\sum_1^\infty \left\{\frac{1}{(x+in)^{14}}+\frac{1}{(x-in)^{14}}\right\}.$$

Letting x have integral values and summing from 1 to ∞ gives

$$\frac{1^{13}}{e^{2\pi}-1}+\frac{2^{13}}{e^{4\pi}-1}+\frac{3^{13}}{e^{6\pi}-1}+\dots$$
$$=\frac{13!}{(2\pi)^{14}}\zeta(14)+\frac{13!}{(2\pi)^{14}}\sum_{1,1}^{\infty,\infty}\left\{\frac{1}{(m+in)^{14}}+\frac{1}{(m-in)^{14}}\right\}.$$

Now
$$\zeta(14)=\frac{(2\pi)^{14}}{14!}\frac{7}{12},$$

and the double summation, changing sign on interchanging m and n, therefore vanishes. Hence

$$\sum_{1}^{\infty}\frac{n^{13}}{e^{2\pi n}-1}=\frac{1}{24}.$$

2. Since
$$\frac{2\pi e^{2\pi x}}{(e^{2\pi x}-1)^2}=\frac{1}{2\pi x^2}-\frac{1}{\pi}\sum_{p=1}^{\infty}\frac{p^2-x^2}{(p^2+x^2)^2},$$

therefore
$$\sum_{q=1}^{\infty}\frac{2\pi e^{2\pi q}}{(e^{2\pi q}-1)^2}=\sum_{q=1}^{\infty}\sum_{n=1}^{\infty}2\pi n e^{-2\pi nq}=\sum_{1}^{\infty}\frac{2\pi n}{e^{2\pi n}-1}$$
$$=\frac{\pi}{12}-\frac{1}{\pi}\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\frac{p^2-q^2}{(p^2+q^2)^2}.$$

To evaluate the double summation, take $P>Q$; then

$$\sum_{p=1}^{P}\sum_{q=1}^{Q}\frac{p^2-q^2}{(p^2+q^2)^2}=\sum_{p=Q+1}^{P}\sum_{q=1}^{Q}\frac{p^2-q^2}{(p^2+q^2)^2}.$$

By the lemma,

$$-f(0)+\int_0^n f(x)\,dx<f(1)+f(2)+\dots+f(n)<\int_0^n f(x)\,dx$$

when $f(x)$ is positive and decreasing, we get

$$0<\frac{Q}{p^2+Q^2}-\sum_{q=1}^{Q}\frac{p^2-q^2}{(p^2+q^2)^2}<\frac{1}{p^2}.$$

Since
$$\lim_{Q\to\infty}\lim_{P\to\infty}\sum_{Q+1}^{P}\frac{1}{p^2}=0,$$

therefore
$$\lim_{Q\to\infty}\lim_{P\to\infty}\sum_{q=Q+1}^{P}\sum_{q=1}^{Q}\frac{p^2-q^2}{(p^2+q^2)^2}=\lim_{Q\to\infty}\sum_{p=Q+1}^{\infty}\frac{Q}{p^2+Q^2}.$$

Again, by the lemma,

$$\tfrac{1}{4}\pi-\frac{1}{2Q}<\sum_{p=Q+1}^{\infty}\frac{Q}{p^2+Q^2}<\tfrac{1}{4}\pi,$$

so that
$$\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\frac{p^2-q^2}{(p^2+q^2)^2}=\tfrac{1}{4}\pi.$$

Hence
$$\sum_{1}^{\infty}\frac{n}{e^{2\pi n}-1}=\frac{1}{24}-\frac{1}{8\pi}.$$

3. Since
$$\coth\pi x=\frac{1}{\pi x}+\frac{2}{\pi}\sum_{1}^{\infty}\frac{x}{x^2+n^2},$$

therefore
$$\sum_{1}^{\infty}\frac{\coth\pi n}{n^7}=\frac{1}{\pi}\zeta(8)+\frac{2}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^6(n^2+m^2)}.$$

Write
$$\begin{aligned}S&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^6(n^2+m^2)}\\&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{n^6m^2}-\frac{1}{n^4m^4}+\frac{1}{n^2m^6}-\frac{1}{m^6(m^2+n^2)}\right)\\&=\zeta(6)\zeta(2)-\zeta(4)\zeta(4)+\zeta(2)\zeta(6)-S.\end{aligned}$$

Hence
$$S=\zeta(2)\zeta(6)-\tfrac{1}{2}\zeta^2(4).$$

Substituting for the ζ functions gives
$$\sum_{1}^{\infty}\frac{\coth\pi n}{n^7}=\frac{19\pi^7}{56700}.$$

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